

THOMAS ERNST

## On the complex $q$ -Appell polynomials

ABSTRACT. The purpose of this article is to generalize the ring of  $q$ -Appell polynomials to the complex case. The formulas for  $q$ -Appell polynomials thus appear again, with similar names, in a purely symmetric way. Since these complex  $q$ -Appell polynomials are also  $q$ -complex analytic functions, we are able to give a first example of the  $q$ -Cauchy–Riemann equations. Similarly, in the spirit of Kim and Ryoo, we can define  $q$ -complex Bernoulli and Euler polynomials. Previously, in order to obtain the  $q$ -Appell polynomial, we would make a  $q$ -addition of the corresponding  $q$ -Appell number with  $x$ . This is now replaced by a  $q$ -addition of the corresponding  $q$ -Appell number with two infinite function sequences  $\mathcal{C}_{\nu,q}(x, y)$  and  $\mathcal{S}_{\nu,q}(x, y)$  for the real and imaginary part of a new so-called  $q$ -complex number appearing in the generating function. Finally, we can prove  $q$ -analogues of the Cauchy–Riemann equations.

This paper is organized as follows: in Section 1, we present the simplest type of  $q$ -complex numbers, which we will later use as function arguments in our new complex  $q$ -Appell polynomials. We remark that there are also other types of  $q$ -complex numbers.

In Section 2, we define complex  $q$ -Appell polynomials, and show that these polynomials obey quite similar rules as  $q$ -Appell polynomials, which always appear in doublets.

In Section 3, we briefly discuss pseudo- $q$ -complex Appell polynomials, which have a slightly different generating function.

---

2010 *Mathematics Subject Classification*. Primary 11B68; Secondary 05A40, 05A30, 30B10.

*Key words and phrases*. Complex  $q$ -Appell polynomials,  $q$ -complex numbers,  $q$ -complex Bernoulli and Euler polynomials,  $q$ -Cauchy–Riemann equations.

In Section 4, we present the two simplest examples of  $q$ -complex Appell polynomials. We remark that many other such polynomials could easily be defined, like in the other papers of the author. The special formulas, which appear in Section 2, are not repeated, for the sake of brevity.

In Section 5, we make a brief conclusion.

### 1. Definition of the $q$ -complex numbers $\mathbb{C}_{\oplus_q}$ .

**Definition 1.** We define the  $q$ -complex numbers  $\mathbb{C}_{\oplus_q}$  as the set

$$\{z = x \oplus_q iy\}, \quad x, y \in \mathbb{R}.$$

The numbers  $x$  and  $y$  are called the  $q$ -real and the  $q$ -imaginary parts of  $z$ , denoted by  $\text{Re}_q z$  and  $\text{Im}_q z$ . A  $q$ -real number is a  $q$ -complex number with  $q$ -imaginary part 0. A  $q$ -imaginary number is a  $q$ -complex number with  $q$ -real part 0.

**Definition 2.** The absolute value of  $z \in \mathbb{C}_{\oplus_q}$  is given by

$$|z| \equiv \sqrt{x^2 + y^2}.$$

The conjugate of  $z \in \mathbb{C}_{\oplus_q}$  is given by

$$\bar{z} \equiv x \ominus_q iy.$$

For each  $z \in \mathbb{C}_{\oplus_q}$ ,  $a \in \mathbb{R}$ , we define a scalar multiplication  $az$ :

$$az \equiv ax \oplus_q iay.$$

We will now define the corresponding operators of  $+$ ,  $-$ ,  $\odot$ ,  $\div$  by considering the set  $(\mathbb{C}_{\oplus_q}, \oplus, \odot_q)$ . The two operations  $\oplus$  and  $\odot_q$  are defined with the help of the following operator  $h$ .

**Definition 3.** The bijection  $h : \mathbb{C} \longrightarrow \mathbb{C}_{\oplus_q}$  maps the complex number  $x+iy$  to  $x \oplus_q iy \in \mathbb{C}_{\oplus_q}$ .

In the following we will use the notation

$$\begin{aligned} z_j &= x_j \oplus_q iy_j, \quad z_j \in \mathbb{C}_{\oplus_q}. \\ \alpha &= \alpha_1 \oplus_q i\alpha_2, \quad \beta = \beta_1 \oplus_q i\beta_2, \quad \gamma = \gamma_1 \oplus_q i\gamma_2, \quad \alpha, \beta, \gamma \in \mathbb{C}_{\oplus_q}. \end{aligned}$$

**Definition 4.** The four binary operations are defined as follows: An addition  $\oplus : \mathbb{C}_{\oplus_q} \times \mathbb{C}_{\oplus_q} \longrightarrow \mathbb{C}_{\oplus_q}$  by

$$z_1 \oplus z_2 \equiv h(h^{-1}z_1 + h^{-1}z_2).$$

A subtraction  $\ominus : \mathbb{C}_{\oplus_q} \times \mathbb{C}_{\oplus_q} \longrightarrow \mathbb{C}_{\oplus_q}$ :

$$z_1 \ominus z_2 \equiv z_1 \oplus (-z_2).$$

A multiplication  $\odot : \mathbb{C}_{\oplus q} \times \mathbb{C}_{\oplus q} \longrightarrow \mathbb{C}_{\oplus q}$ :

$$z_1 \odot z_2 \equiv h(h^{-1}z_1 \cdot h^{-1}z_2).$$

A division  $\oslash : \mathbb{C}_{\oplus q} \times \mathbb{C}_{\oplus q} \longrightarrow \mathbb{C}_{\oplus q}$ :

$$z_1 \oslash z_2 \equiv h \left( \frac{h^{-1}z_1}{h^{-1}z_2} \right).$$

The multiplication and division are also given by

$$\begin{aligned} z_1 \odot z_2 &= (x_1x_2 - y_1y_2) \oplus_q i(x_1y_2 + x_2y_1), \\ z_1 \oslash z_2 &= \frac{1}{x_2^2 + y_2^2} ((x_1x_2 + y_1y_2) \oplus_q i(x_2y_1 - x_1y_2)). \end{aligned}$$

Then

$$(z_1 \oslash z_2) \odot z_2 = z_1.$$

We decide to have the same priority for these operations as usual, i.e., products are computed before sums (additions) etc. We also agree to sometimes abbreviate the  $\odot$  by juxtaposition.

For purposes which will soon become evident, we limit ourselves to formal power series.

**Definition 5.** We define the complex  $q$ -derivative as

$$D_{\oplus} f(z) \equiv \lim_{\delta z \rightarrow 0_{\oplus q}} \frac{f(z \oplus \delta z) - f(z)}{(\delta z)^1}, \quad f \in \mathbb{C}[[z]].$$

Powers of  $D_{\oplus}$  are denoted by  $D_{\oplus}^m$ .

The function  $f(z)$  is called  $q$ -holomorphic if and only if the complex  $q$ -derivative  $D_{\oplus} f(z)$  exists.

**Theorem 1.1.** *Formula for the complex  $q$ -derivative for functions of  $q$ -complex numbers.*

$$D_{\oplus} \sum_{k=0}^{\infty} a_k z^k = \sum_{k=1}^{\infty} a_k \{k\}_q z^{k-1}.$$

**Proof.** We denote the fact that  $h$  is an isomorphism by  $\star$ . Put  $f(z) = \sum_{k=0}^{\infty} a_k z^k$ . Then

$$\begin{aligned} \frac{f(z \oplus \delta z) - f(z)}{(\delta z)^1} &= \frac{f((x \oplus_q iy) \oplus (\delta x \oplus_q i\delta y)) - f(x \oplus_q iy)}{(\delta z)^1} \\ &\stackrel{\text{by } \star}{=} \frac{f((x + \delta x) \oplus_q i(y + \delta y)) - f(x \oplus_q iy)}{(\delta z)^1} \\ &= \frac{\sum_{k=0}^{\infty} a_k ((x + \delta x) \oplus_q i(y + \delta y))^k - \sum_{k=0}^{\infty} a_k (x \oplus_q iy)^k}{(\delta z)^1} \end{aligned}$$

$$\begin{aligned}
&= \frac{\sum_{k=0}^{\infty} a_k \sum_{m=0}^k \binom{k}{m}_q (i(y + \delta y))^{k-m} (x + \delta x)^m}{(\delta z)^1} \\
&\quad - \frac{\sum_{k=0}^{\infty} a_k \sum_{m=0}^k \binom{k}{m}_q (iy)^{k-m} x^m}{(\delta z)^1} \\
&= \frac{1}{(\delta z)^1} \left[ \sum_{k=0}^{\infty} a_k \sum_{m=1}^k \binom{k}{m}_q \sum_{l=0}^m \binom{m}{l} x^{m-l} \delta x^l i^{k-m} \sum_{n=0}^{k-m} \binom{k-m}{n} \delta y^n y^{k-m-n} \right] \\
&= \frac{1}{(\delta z)^1} \sum_{k=0}^{\infty} a_k \sum_{m=1}^k \binom{k}{m}_q z^{k-m} (\delta z)^m.
\end{aligned}$$

The result now follows by letting  $\lim_{\delta z \rightarrow 0_{\oplus q}}$ .  $\square$

**Theorem 1.2.**

$$D_{\oplus} f(z) = D_{q,x} u(x, y) + i D_{q,x} v(x, y).$$

**Proof.**

$$\begin{aligned}
D_{q,x} u(x, y) + i D_{q,x} v(x, y) &= D_{q,x} \sum_{k=0}^{\infty} a_k \sum_{l=0}^k \binom{k}{l}_q x^l (iy)^{k-l} \\
&= \sum_{k=0}^{\infty} a_k \sum_{l=1}^k \binom{k}{l}_q \{l\}_q x^{l-1} (iy)^{k-l} = D_{\oplus} f(z). \quad \square
\end{aligned}$$

**2. Extension of  $q$ -Appell polynomials to complex  $q$ -Appell polynomials.** We will now define the complex  $q$ -Appell polynomials. Throughout, we assume that  $z = x \oplus_q iy$ , where we can use both of the previously defined  $q$ -complex numbers. In the beginning, we use the numbers  $\mathbb{C}_{\oplus q}$ . For the notation, we refer to [4].

**Definition 6** (A  $q$ -analogue of [5, (3)]). For every power series  $f_n(t)$ , with  $f_n(0) \neq 0$ , the cosine- $q$ -Appell polynomials  $\mathcal{A}_{\nu,q}^{(c,n)}(x, y)$  have the following generating function:

$$(1) \quad f_n(t) E_q(xt) \text{Cos}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \mathcal{A}_{\nu,q}^{(c,n)}(x, y).$$

The sine- $q$ -Appell polynomials  $\mathcal{A}_q(s, n)$  of degree  $\nu$  and order  $n$  have the following generating function:

$$(2) \quad f_n(t) E_q(xt) \text{Sin}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \mathcal{A}_{\nu,q}^{(s,n)}(x, y).$$

By putting  $x = y = 0$ , we have again (see [2, 4.105])

$$f_n(t) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \Phi_{\nu,q}^{(n)},$$

where  $\Phi_{\nu,q}^{(n)}$  are the  $q$ -Appell numbers. In the following, when a formula with  $\mathcal{A}_{\nu,q}^{(c,n)}(x)$  is given, without a similar formula with  $\mathcal{A}_{\nu,q}^{(s,n)}(x)$ , we always assume that  $c \equiv c \vee s$ .

It will be convenient to fix the value for  $n = 0$  and  $n = 1$ :

$$\mathcal{A}_{\nu,q}^{(c,1)}(x) \equiv \mathcal{A}_{\nu,q}^{(c)}(x); \mathcal{A}_{\nu,q}^{(c,0)} \equiv \Phi_{\nu,q}^{(0)} \equiv 0.$$

The following formula is the same as [2, 4.107]:

$$D_q \mathcal{A}_{\nu,q}^{(c,n)}(x) = \{\nu\}_q \mathcal{A}_{\nu-1,q}^{(c,n)}(x).$$

A  $q$ -analogue of [5, (3)] expresses the  $q$ -Appell polynomial of  $z$  as the sum of the cosine and sine- $q$ -Appell polynomials.

$$(3) \quad \Phi_{\nu,q}^{(n)}(z) = \mathcal{A}_{\nu,q}^{(c,n)}(x, y) + i \mathcal{A}_{\nu,q}^{(s,n)}(x, y).$$

Then we have generating functions for  $q$ -Appell polynomial of  $z$  and  $\bar{z}$ :

$$(4) \quad f_n(t) E_q(z t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi_{\nu,q}^{(n)}(z) = f_n(t) E_q(x t) (\text{Cos}_q(y t) + i \text{Sin}_q(y t)),$$

$$(5) \quad f_n(t) E_q(\bar{z} t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi_{\nu,q}^{(n)}(\bar{z}) = f_n(t) E_q(x t) (\text{Cos}_q(y t) - i \text{Sin}_q(y t)).$$

Addition and subtraction of formulas (4) and (5) give a  $q$ -analogue of [5, p. 3]:

$$\begin{aligned} f_n(t) E_q(x t) \text{Cos}_q(y t) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \frac{\Phi_{\nu,q}^{(n)}(z) + \Phi_{\nu,q}^{(n)}(\bar{z})}{2}, \\ f_n(t) E_q(x t) \text{Sin}_q(y t) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \frac{\Phi_{\nu,q}^{(n)}(z) - \Phi_{\nu,q}^{(n)}(\bar{z})}{2i}, \\ \mathcal{A}_{\nu,q}^{(c,n)}(x, y) &= \frac{\Phi_{\nu,q}^{(n)}(z) + \Phi_{\nu,q}^{(n)}(\bar{z})}{2}, \\ \mathcal{A}_{\nu,q}^{(s,n)}(x, y) &= \frac{\Phi_{\nu,q}^{(n)}(z) - \Phi_{\nu,q}^{(n)}(\bar{z})}{2i}. \end{aligned}$$

We obtain the two  $q$ -Taylor formulas, a  $q$ -analogue of [5, (ii), p. 3].

**Theorem 2.1.**

$$\begin{aligned} \Phi_{\nu,q}^{(n)}(x \oplus_q i y) &= \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi_{\nu-k,q}^{(n)}(x) (i y)^k, \\ \Phi_{\nu,q}^{(n)}(x \boxplus_q i y) &= \sum_{k=0}^{\nu} \binom{\nu}{k}_q q^{\binom{k}{2}} \Phi_{\nu-k,q}^{(n)}(x) (i y)^k. \end{aligned}$$

The first formula implies the symbolic equality:

**Theorem 2.2.**

$$\Phi_{\nu,q}^{(n)}(z) \doteq (\Phi_q^{(n)}(x) \oplus_q iy)^\nu.$$

We collect some obvious facts about the  $q$ -complex Appell polynomials in a theorem.

**Theorem 2.3** (A  $q$ -analogue of [5, p. 3]).

$$\Phi_{\nu,q}^{(n)}(\bar{z}) = \overline{\Phi_{\nu,q}^{(n)}(z)}.$$

$\mathcal{A}_{\nu,q}^{(c,n)}(x, y)$  is an even function of  $y$ .  $\mathcal{A}_{\nu,q}^{(s,n)}(x, y)$  is an odd function of  $y$ .

$$(6) \quad \Phi_{\nu,q}^{(n)}(z)\Phi_{\nu,q}^{(n)}(\bar{z}) = \left(\mathcal{A}_{\nu,q}^{(c,n)}(x, y)\right)^2 + \left(\mathcal{A}_{\nu,q}^{(s,n)}(x, y)\right)^2.$$

$$(7) \quad \begin{aligned} \Phi_{\nu,q}^{(n)}(z)\Phi_{\mu,q}^{(n)}(\bar{z}) &= \left(\mathcal{A}_{\nu,q}^{(c,n)}(x, y)\mathcal{A}_{\mu,q}^{(c,n)}(x, y) + \mathcal{A}_{\nu,q}^{(s,n)}(x, y)\mathcal{A}_{\mu,q}^{(s,n)}(x, y)\right) \\ &+ i \left(\mathcal{A}_{\nu,q}^{(s,n)}(x, y)\mathcal{A}_{\mu,q}^{(c,n)}(x, y) - \mathcal{A}_{\nu,q}^{(c,n)}(x, y)\mathcal{A}_{\mu,q}^{(s,n)}(x, y)\right). \end{aligned}$$

**Proof.** To prove (6), use (3).  $\square$

**Lemma 2.4** (A  $q$ -analogue of [5, (4) and (5)]). *The cosine- $q$ -Appell polynomials and the sine- $q$ -Appell polynomials can be expressed as sums of the corresponding  $q$ -Appell polynomials with coefficients powers of  $y$ .*

$$(8) \quad \mathcal{A}_{\nu,q}^{(c,n)}(x, y) = \sum_{k=0}^{\lfloor \frac{\nu}{2} \rfloor} \binom{\nu}{2k}_q (-1)^k y^{2k} \Phi_{\nu-2k,q}^{(n)}(x),$$

$$(9) \quad \mathcal{A}_{\nu,q}^{(s,n)}(x, y) = \sum_{k=0}^{\lfloor \frac{\nu-1}{2} \rfloor} \binom{\nu}{2k+1}_q (-1)^k y^{2k+1} \Phi_{\nu-2k-1,q}^{(n)}(x).$$

**Proof.** To prove (8), use generating function (1) together with the  $q$ -Euler formula and change summation index.  $\square$

**Lemma 2.5** (A  $q$ -analogue of [5, Lemma 2, p. 4]). *Formulas for sums of products of  $q$ -Appell polynomials with mixed function arguments  $z$  and  $\bar{z}$ .*

$$(10) \quad \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi_{k,q}^{(n)}(z)\Phi_{\nu-k,q}^{(n)}(\bar{z}) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi_{k,q}^{(n)}(x)\Phi_{\nu-k,q}^{(n)}(x),$$

$$(11) \quad \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi_{k,q}^{(n)}(z)\Phi_{\nu-k,q}^{(n)}(-\bar{z}) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi_{k,q}^{(n)}(iy)\Phi_{\nu-k,q}^{(n)}(iy).$$

**Proof.** To prove (10), use generating function (1).  $\square$

The following two functions  $\mathcal{C}_{\nu,q}(x, y)$  and  $\mathcal{S}_{\nu,q}(x, y)$  replace  $q$ -addition of  $x$  with  $q$ -Appell numbers.

**Definition 7** (A  $q$ -analogue of [5, p. 5]).

$$(12) \quad E_q(xt) \operatorname{Cos}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathcal{C}_{\nu,q}(x, y),$$

$$(13) \quad E_q(xt) \operatorname{Sin}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathcal{S}_{\nu,q}(x, y).$$

We find that (compare with [5, (9), (10)])

$$\begin{aligned} \mathcal{A}_{\nu,q}^{(c,n)}(x, y) &= (\Phi_{\nu,q}^{(n)} \oplus_q \mathcal{C}_{\nu,q}(x, y))^\nu, \\ \mathcal{A}_{\nu,q}^{(s,n)}(x, y) &= (\Phi_{\nu,q}^{(n)} \oplus_q \mathcal{S}_{\nu,q}(x, y))^\nu. \end{aligned}$$

A  $q$ -analogue of [5, p. 5]; explicit expressions for  $\mathcal{C}_{\nu,q}(x, y)$  and  $\mathcal{S}_{\nu,q}(x, y)$ :

$$\begin{aligned} \mathcal{C}_{\nu,q}(z) \equiv \mathcal{C}_{\nu,q}(x, y) &= \sum_{k=0}^{\lfloor \frac{\nu}{2} \rfloor} \binom{\nu}{2k}_q (-1)^k y^{2k} x^{\nu-2k}, \\ \mathcal{S}_{\nu,q}(z) \equiv \mathcal{S}_{\nu,q}(x, y) &= \sum_{k=0}^{\lfloor \frac{\nu-1}{2} \rfloor} \binom{\nu}{2k+1}_q (-1)^k y^{2k+1} x^{\nu-2k-1}. \end{aligned}$$

**Theorem 2.6** (A  $q$ -analogue of [5, p. 5]). *Addition formulas for  $\mathcal{C}_{\nu,q}(z)$  and  $\mathcal{S}_{\nu,q}(z)$ . Let  $\oplus_q$  denote  $\oplus_q \vee \ominus_q$ , corresponding to the two  $\pm$ . Then*

$$(14) \quad \mathcal{C}_{n,q}(z_1 \oplus_q z_2) = \sum_{k=0}^n \binom{n}{k}_q (\mathcal{C}_{n-k,q}(z_1) \mathcal{C}_{k,q}(\pm z_2) - \mathcal{S}_{n-k,q}(z_1) \mathcal{S}_{k,q}(\pm z_2)),$$

$$(15) \quad \mathcal{S}_{n,q}(z_1 \oplus_q z_2) = \sum_{k=0}^n \binom{n}{k}_q (\mathcal{S}_{n-k,q}(z_1) \mathcal{C}_{k,q}(\pm z_2) + \mathcal{C}_{n-k,q}(z_1) \mathcal{S}_{k,q}(\pm z_2)).$$

**Proof.** We find that

$$\begin{aligned} E_q((x_1 \oplus_q x_2)t) \operatorname{Cos}_q((y_1 \oplus_q y_2)t) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathcal{C}_{\nu,q}(z_1 \oplus_q z_2), \\ E_q((x_1 \oplus_q x_2)t) \operatorname{Sin}_q((y_1 \oplus_q y_2)t) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathcal{S}_{\nu,q}(z_1 \oplus_q z_2). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &E_q((x_1 \oplus_q x_2)t) \operatorname{Cos}_q((y_1 \oplus_q y_2)t) \\ &= E_q((x_1 \oplus_q x_2)t) (\operatorname{Cos}_q(y_1t) \operatorname{Cos}_q(y_2t) \mp \operatorname{Sin}_q(y_1t) \operatorname{Sin}_q(y_2t)) \\ &= E_q((x_1t) \operatorname{Cos}_q(y_1t)) E_q((\pm x_2t) \operatorname{Cos}_q(y_2t)) \\ &\quad \mp E_q((x_1t) \operatorname{Sin}_q(y_1t)) E_q((\pm x_2t) \operatorname{Sin}_q(y_2t)) \\ &= E_q((x_1t) \operatorname{Cos}_q(y_1t)) E_q((\pm x_2t) \operatorname{Cos}_q(\pm y_2t)) \\ &\quad - E_q((x_1t) \operatorname{Sin}_q(y_1t)) E_q((\pm x_2t) \operatorname{Sin}_q(\pm y_2t)) \end{aligned}$$

$$\begin{aligned}
& \stackrel{\text{by (12), (13)}}{=} \left( \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathcal{C}_{\nu,q}(z_1) \right) \left( \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathcal{C}_{\nu,q}(\pm z_2) \right) \\
& - \left( \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathcal{S}_{\nu,q}(z_1) \right) \left( \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathcal{S}_{\nu,q}(\pm z_2) \right) \\
& = \sum_{\nu=0}^{\infty} ((\mathcal{C}_q(z_1) \oplus_q \mathcal{C}_q(z_2))^\nu - (\mathcal{S}_q(z_1) \oplus_q \mathcal{S}_q(z_2))^\nu) \frac{t^\nu}{\{\nu\}_q!},
\end{aligned}$$

which proves (14). Formula (15) is proved in a similar way.  $\square$

We can now prove  $q$ -analogues of the Cauchy–Riemann equations for  $q$ -complex Appell polynomials.

**Lemma 2.7** (A  $q$ -analogue of [5, p. 6]).

$$(16) \quad D_{q,x} \mathcal{A}_{\nu,q}^{(c,n)}(x, y) = \{\nu\}_q \mathcal{A}_{\nu-1,q}^{(c,n)}(x, y).$$

$$(17) \quad D_{q,x} \mathcal{A}_{\nu,q}^{(s,n)}(x, y) = \{\nu\}_q \mathcal{A}_{\nu-1,q}^{(s,n)}(x, y).$$

$$(18) \quad D_{q,y} \mathcal{A}_{\nu,q}^{(c,n)}(x, y) = -\{\nu\}_q \mathcal{A}_{\nu-1,q}^{(s,n)}(x, y).$$

$$(19) \quad D_{q,y} \mathcal{A}_{\nu,q}^{(s,n)}(x, y) = \{\nu\}_q \mathcal{A}_{\nu-1,q}^{(c,n)}(x, y).$$

**Proof.** Use the generating function.  $\square$

**Theorem 2.8** ( $q$ -analogues of the Cauchy–Riemann equations [1, p. 54]).  
for  $q$ -complex Appell polynomials. Let our  $q$ -complex function be  $\Phi_{\nu,q}^{(n)}$ , with  $q$ -real and  $q$ -imaginary parts given by (3). Then

$$\begin{aligned}
D_{q,x} \mathcal{A}_{\nu,q}^{(c,n)}(z) &= D_{q,y} \mathcal{A}_{\nu,q}^{(s,n)}(z), \\
D_{q,y} \mathcal{A}_{\nu,q}^{(c,n)}(z) &= -D_{q,x} \mathcal{A}_{\nu,q}^{(s,n)}(z).
\end{aligned}$$

**Proof.** Equate formulas (16), (19), and formulas (17), (18), respectively.  $\square$

### 3. Pseudo- $q$ -complex Appell polynomials.

**Definition 8.** For every power series  $f_n(t)$  given by (1) and (2), the pseudo-cosine and pseudosine- $q$ -Appell polynomials  $\mathcal{A}_\nu^{(q,c,n)}$  and  $\mathcal{A}_\nu^{(q,s,n)}$  of degree  $\nu$  and order  $n$  have the following generating functions:

$$\begin{aligned}
f_n(t) E_{\frac{1}{q}}(xt) \text{Cos}_q(yt) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathcal{A}_\nu^{(q,c,n)}(x, y), \\
f_n(t) E_{\frac{1}{q}}(xt) \text{Sin}_q(yt) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathcal{A}_\nu^{(q,s,n)}(x, y).
\end{aligned}$$



Now, for convenience, we fix the value for  $n = 1$ :

$$\begin{aligned}\mathcal{A}_\nu^{(q,c)}(x, y) &\equiv \mathcal{A}_\nu^{(q,c,1)}(x, y), \\ \mathcal{A}_\nu^{(q,s)}(x, y) &\equiv \mathcal{A}_\nu^{(q,s,1)}(x, y).\end{aligned}$$

We have

**Theorem 3.1.**

$$(20) \quad D_q \mathcal{A}_\nu^{(q,c,n)}(x) = \{\nu\}_q \mathcal{A}_{\nu-1}^{(q,c,n)}(qx).$$

$$(21) \quad D_q \mathcal{A}_\nu^{(q,s,n)}(x) = \{\nu\}_q \mathcal{A}_{\nu-1}^{(q,s,n)}(qx).$$

We get the following two  $q$ -Taylor formulas:

**Theorem 3.2.**

$$\begin{aligned}\mathcal{A}_\nu^{(n;q)}(x \oplus_q y) &= \sum_{k=0}^{\nu} \binom{\nu}{k}_q q^{\binom{k}{2}} \mathcal{A}_\nu^{(n;q)}(q^k x) y^k, \\ \mathcal{A}_\nu^{(n;q)}(x \boxplus_q y) &= \sum_{k=0}^{\nu} \binom{\nu}{k}_q q^{2\binom{k}{2}} \mathcal{A}_\nu^{(n;q)}(q^k x) y^k.\end{aligned}$$

**Proof.** Use formula (20). □

As a prerequisite of the next section, we extend the following formulas from [2]. The two operators  $\Delta_{\text{NWA},q}$  and  $\nabla_{\text{NWA},q}$  always refer to the variable  $x$ .

**Theorem 3.3.**

$$\begin{aligned}(\mathbb{E}_q(t) - 1) f_n(t) \mathbb{E}_q(xt) \text{Cos}_q(yt) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Delta_{\text{NWA},q} \mathcal{A}_{\nu,q}^{(c,n)}(x, y), \\ (\mathbb{E}_q(t) - 1) f_n(t) \mathbb{E}_q(xt) \text{Sin}_q(yt) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Delta_{\text{NWA},q} \mathcal{A}_{\nu,q}^{(s,n)}(x, y).\end{aligned}$$

**Theorem 3.4.**

$$\begin{aligned}\frac{(\mathbb{E}_q(t) + 1)}{2} f_n(t) \mathbb{E}_q(xt) \text{Cos}_q(yt) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \nabla_{\text{NWA},q} \mathcal{A}_{\nu,q}^{(c,n)}(x, y), \\ \frac{(\mathbb{E}_q(t) + 1)}{2} f_n(t) \mathbb{E}_q(xt) \text{Sin}_q(yt) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \nabla_{\text{NWA},q} \mathcal{A}_{\nu,q}^{(s,n)}(x, y).\end{aligned}$$

**Theorem 3.5.**

$$(22) \quad (\mathbb{E}_{\frac{1}{q}}(t) - 1) f_n(t) \mathbb{E}_q(xt) \text{Cos}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Delta_{\text{JHC},q} \mathcal{A}_{\nu,q}^{(c,n)}(x, y),$$

$$(23) \quad (\mathbb{E}_{\frac{1}{q}}(t) - 1) f_n(t) \mathbb{E}_q(xt) \text{Sin}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Delta_{\text{JHC},q} \mathcal{A}_{\nu,q}^{(s,n)}(x, y).$$

**Theorem 3.6.**

$$(24) \quad \frac{(\mathbb{E}_{\frac{1}{q}}(t) + 1)}{2} f_n(t) \mathbb{E}_q(xt) \text{Cos}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \nabla_{\text{JHC},q} \mathcal{A}_{\nu,q}^{(c,n)}(x, y),$$

$$(25) \quad \frac{(\mathbb{E}_{\frac{1}{q}}(t) + 1)}{2} f_n(t) \mathbb{E}_q(xt) \text{Sin}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \nabla_{\text{JHC},q} \mathcal{A}_{\nu,q}^{(s,n)}(x, y).$$

**4. Special complex  $q$ -Appell polynomials.** A special case of the  $\mathcal{A}_q$  polynomials are the two complex  $\beta_q$ -polynomials of degree  $\nu$  and order  $n$ , which are obtained by putting  $f_n(t) = \frac{t^n g(t)}{(\mathbb{E}_q(t) - 1)^n}$  in (1) and (2):

**Definition 9.**

$$\frac{t^n g(t)}{(\mathbb{E}_q(t) - 1)^n} \mathbb{E}_q(xt) \text{Cos}_q(yt) \equiv \sum_{\nu=0}^{\infty} \frac{t^\nu \beta_{\nu,q}^{(c,n)}(x, y)}{\{\nu\}_q!},$$

$$\frac{t^n g(t)}{(\mathbb{E}_q(t) - 1)^n} \mathbb{E}_q(xt) \text{Sin}_q(yt) \equiv \sum_{\nu=0}^{\infty} \frac{t^\nu \beta_{\nu,q}^{(s,n)}(x, y)}{\{\nu\}_q!}.$$

Similar to [2, 4.119], we find that

**Theorem 4.1** (A  $q$ -analogue of [4, p. 10]).

$$\Delta_{\text{NWA},q} \beta_{\nu,q}^{(c,n)}(x, y) = \{\nu\}_q \beta_{\nu-1,q}^{(c,n-1)}(x, y) = D_{q,x} \beta_{\nu,q}^{(c,n-1)}(x, y),$$

$$\Delta_{\text{NWA},q} \beta_{\nu,q}^{(s,n)}(x, y) = \{\nu\}_q \beta_{\nu-1,q}^{(s,n-1)}(x, y) = D_{q,x} \beta_{\nu,q}^{(s,n-1)}(x, y).$$

**Definition 10** (A  $q$ -analogue of [4, (32), (33)]). The generating functions for  $B_{\text{NWA},\nu,q}^{(c,n)}(x, y)$  and  $B_{\text{NWA},\nu,q}^{(s,n)}(x, y)$ :

$$\frac{t^n}{(\mathbb{E}_q(t) - 1)^n} \mathbb{E}_q(xt) \text{Cos}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^\nu B_{\text{NWA},\nu,q}^{(c,n)}(x, y)}{\{\nu\}_q!}, \quad |t| < 2\pi.$$

$$\frac{t^n}{(\mathbb{E}_q(t) - 1)^n} \mathbb{E}_q(xt) \text{Sin}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^\nu B_{\text{NWA},\nu,q}^{(s,n)}(x, y)}{\{\nu\}_q!}, \quad |t| < 2\pi.$$

The Ward  $q$ -Bernoulli numbers from [2, p. 118] are used here as well to form the new  $q$ -Bernoulli polynomials.

A special case of complex  $\mathcal{A}_q$  polynomials are the two complex  $\gamma_q$ -polynomials of degree  $\nu$  and order  $n$ , which are obtained by putting  $f_n(t) = \frac{t^n g(t)}{(\mathbb{E}_{\frac{1}{q}}(t) - 1)^n}$  in (1) and (2):

**Definition 11.**

$$\frac{t^n g(t)}{(\mathbb{E}_{\frac{1}{q}}(t) - 1)^n} \mathbb{E}_q(xt) \text{Cos}_q(yt) \equiv \sum_{\nu=0}^{\infty} \frac{t^\nu \gamma_{\nu,q}^{(c,n)}(x, y)}{\{\nu\}_q!},$$

$$\frac{t^n g(t)}{(\mathbb{E}_{\frac{1}{q}}(t) - 1)^n} \mathbb{E}_q(xt) \text{Sin}_q(yt) \equiv \sum_{\nu=0}^{\infty} \frac{t^\nu \gamma_{\nu,q}^{(s,n)}(x,y)}{\{\nu\}_q!}.$$

Similar to [2, 4.159], we find that

**Theorem 4.2.**

$$\Delta_{\text{JHC},q} \gamma_{\nu,q}^{(c,n)}(x,y) = \{\nu\}_q \gamma_{\nu-1,q}^{(c,n-1)}(x,y) = \mathbb{D}_{q,x} \gamma_{\nu,q}^{(c,n-1)}(x,y).$$

**Proof.** Use (22).  $\square$

A special case of complex  $\gamma_q$  polynomials are the second generalized complex  $q$ -Bernoulli polynomials  $\mathbb{B}_{\text{JHC},\nu,q}^{(c,n)}(x,y)$  of degree  $\nu$  and order  $n$ .

**Definition 12** (Another  $q$ -analogue of [4, (32), (33)]). The generating functions for  $\mathbb{B}_{\text{JHC},\nu,q}^{(c,n)}(x)$  and  $\mathbb{B}_{\text{JHC},\nu,q}^{(s,n)}(x)$  are

$$\begin{aligned} \frac{t^n}{(\mathbb{E}_{\frac{1}{q}}(t) - 1)^n} \mathbb{E}_q(xt) \text{Cos}_q(yt) &= \sum_{\nu=0}^{\infty} \frac{t^\nu \mathbb{B}_{\text{JHC},\nu,q}^{(c,n)}(x,y)}{\{\nu\}_q!}, \quad |t| < 2\pi, \\ \frac{t^n}{(\mathbb{E}_{\frac{1}{q}}(t) - 1)^n} \mathbb{E}_q(xt) \text{Sin}_q(yt) &= \sum_{\nu=0}^{\infty} \frac{t^\nu \mathbb{B}_{\text{JHC},\nu,q}^{(s,n)}(x,y)}{\{\nu\}_q!}, \quad |t| < 2\pi. \end{aligned}$$

We will now define  $q$ -Appell polynomials with a similar character as the previous ones. A special case of the  $\mathcal{A}_q$  polynomials are the  $\eta_q$  polynomials of order  $n$ , which are obtained by putting  $f_n(t) = \frac{g(t)2^n}{(\mathbb{E}_q(t)+1)^n}$  in (1) and (2).

**Definition 13.**

$$\begin{aligned} \frac{2^n}{(\mathbb{E}_q(t) + 1)^n} g(t) \mathbb{E}_q(xt) \text{Cos}_q(yt) &= \sum_{\nu=0}^{\infty} \frac{t^\nu \eta_{\nu,q}^{(c,n)}(x,y)}{\{\nu\}_q!}, \\ \frac{2^n}{(\mathbb{E}_q(t) + 1)^n} g(t) \mathbb{E}_q(xt) \text{Sin}_q(yt) &= \sum_{\nu=0}^{\infty} \frac{t^\nu \eta_{\nu,q}^{(s,n)}(x,y)}{\{\nu\}_q!}. \end{aligned}$$

We get

$$\nabla_{\text{NWA},q} \eta_{\nu,q}^{(c,n)}(x,y) = \eta_{\nu,q}^{(c,n-1)}(x,y).$$

We will now define the first  $q$ -Euler polynomials, a special case of the  $\eta_q$  polynomials.

**Definition 14** (A  $q$ -analogue of [4, (12), (13)]). The generating function for the first  $q$ -Euler polynomials of degree  $\nu$  and order  $n$ ,  $\mathbb{F}_{\text{NWA},\nu,q}^{(c,n)}(x,y)$  is

$$\begin{aligned} \frac{2^n \mathbb{E}_q(xt)}{(\mathbb{E}_q(t) + 1)^n} \text{Cos}_q(yt) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathbb{F}_{\text{NWA},\nu,q}^{(c,n)}(x,y), \quad |t| < \pi, \\ \frac{2^n \mathbb{E}_q(xt)}{(\mathbb{E}_q(t) + 1)^n} \text{Sin}_q(yt) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathbb{F}_{\text{NWA},\nu,q}^{(s,n)}(x,y), \quad |t| < \pi. \end{aligned}$$

**Theorem 4.3** (A  $q$ -analogue of [4, (22)]).

$$\sum_{k=0}^{\nu} \binom{\nu}{k}_q F_{\text{NWA},\nu-k,q}^{(c,n)}(x, y) + F_{\text{NWA},\nu,q}^{(c,n)}(x, y) = 2F_{\text{NWA},\nu,q}^{(c,n-1)}(x, y).$$

A special case of  $\mathcal{A}_q$  polynomials are the  $\theta_q$  polynomials of order  $n$ , which are obtained by putting  $f_n(t) = \frac{g(t)2^n}{(\mathbb{E}_{\frac{1}{q}}(t)+1)^n}$  in (1) and (2).

**Definition 15.**

$$\frac{2^n}{(\mathbb{E}_{\frac{1}{q}}(t)+1)^n} g(t) E_q(xt) \text{Cos}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \theta_{\nu,q}^{(c,n)}(x, y)}{\{\nu\}_q!},$$

$$\frac{2^n}{(\mathbb{E}_{\frac{1}{q}}(t)+1)^n} g(t) E_q(xt) \text{Sin}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \theta_{\nu,q}^{(s,n)}(x, y)}{\{\nu\}_q!}.$$

By (24) we obtain

$$\nabla_{\text{JHC},q} \theta_{\nu,q}^{(n)}(x, y) = \theta_{\nu,q}^{(n-1)}(x, y).$$

We will now define the second  $q$ -Euler polynomials, a special case of the  $\theta_q$  polynomials.

**Definition 16** (Another  $q$ -analogue of [4, (12), (13)]). The generating function for the second  $q$ -Euler polynomials of degree  $\nu$  and order  $n$ ,  $F_{\text{JHC},\nu,q}^{(c,n)}(x, y)$  is

$$\frac{2^n E_q(xt)}{(\mathbb{E}_{\frac{1}{q}}(t)+1)^n} \text{Cos}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} F_{\text{JHC},\nu,q}^{(c,n)}(x, y), \quad |t| < \pi,$$

$$\frac{2^n E_q(xt)}{(\mathbb{E}_{\frac{1}{q}}(t)+1)^n} \text{Sin}_q(yt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} F_{\text{JHC},\nu,q}^{(s,n)}(x, y), \quad |t| < \pi.$$

The proofs of the following four complementary argument formulas are made with the generating function.

**Theorem 4.4** (A  $q$ -analogue of [5, p. 9]).

$$B_{\text{JHC},\nu,q}^{(c)}(x, y) = (-1)^\nu B_{\text{NWA},\nu,q}^{(c)}(1 \ominus_q x, y).$$

**Theorem 4.5** (A  $q$ -analogue of [5, p. 5]).

$$F_{\text{JHC},\nu,q}^{(c)}(x, y) = (-1)^\nu F_{\text{NWA},\nu,q}^{(c)}(1 \ominus_q x, y).$$

**5. Conclusion.** We have introduced a basis for a further investigation of  $q$ -complex numbers, which will appear in another paper. These numbers are used as function arguments in formal power series of one or many variables, not just polynomials. Our proof of the  $q$ -Cauchy–Riemann equations follows [1, p. 54]. It is not unlikely that the  $q$ -Cauchy–Riemann equations can be extended to higher dimensions, like in [6].

## REFERENCES

- [1] Brinck, I., Persson, A., *Elementär teori för analytiska funktioner* (Swedish) (Elementary theory for analytic functions), Lund, 1979.
- [2] Ernst, T., *A Comprehensive Treatment of  $q$ -calculus*, Birkhäuser, Basel, 2012.
- [3] Ernst T., *A new semantics for special functions*, to appear.
- [4] Kim, T., Ryoo, C. S., *Some identities for Euler and Bernoulli polynomials and their zeros*, *Axioms* **7** (3), 56 (2018), pp. 19.
- [5] Kim, D., *A note on the degenerate type of complex Appell polynomials*, *Symmetry* **11** (11), 1339 (2019), pp. 14.
- [6] Range, R., *Holomorphic Functions and Integral Representations in Several Complex Variables*, Springer-Verlag, New York, 1986.

Thomas Ernst  
Department of Mathematics  
Uppsala University  
P.O. Box 480, SE-751 06 Uppsala  
Sweden  
e-mail: [thomas@math.uu.se](mailto:thomas@math.uu.se)

Received November 11, 2019