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Reverse and improved inequalities for operator monotone functions

Abstract. In this paper we provide several refinements and reverse operator inequalities for operator monotone functions in Hilbert spaces. We also obtain refinements and a reverse of Löwner–Heinz celebrated inequality that holds in the case of power function.

1. Introduction. Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function $f(t)$ on $[0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B \geq 0$, which is defined as $A - B \geq 0$.

In 1934, K. Löwner [6] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144–145]:

Theorem 1. *A function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{ts}{t+s} dm(s) \tag{1.1}$$

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where $b \geq 0$ *and* m *is a positive measure on* $[0, \infty)$ *such that*

$$\int_0^\infty \frac{s}{1+s} dm(s) < \infty.$$

We recall the important fact proved by Löwner and Heinz which states that the power function $f : [0, \infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0, 1]$, see [5].

Let $f : (0, \infty) \to (0, \infty)$ be a continuous function. It is known that $f(t)$ is operator monotone if and only if $g(t) = t/f(t) =: f^*(t)$ is also operator monotone, see for instance [3] or [7].

Consider the family of functions defined on $(0, \infty)$ by

$$f_p(t) := \frac{p-1}{p} \left(\frac{t^p - 1}{t^{p-1} - 1} \right)$$

if $p \in [-1, 2] \setminus \{0, 1\}$ and

$$f_0(t) := \frac{t}{1-t} \ln t,$$

$$f_1(t) := \frac{t-1}{\ln t} \text{ (logarithmic mean).}$$

We also have the functions of interest:

$$f_{-1}(t) = \frac{2t}{1+t} \text{ (harmonic mean)}, \ f_{1/2}(t) = \sqrt{t} \text{ (geometric mean).}$$

In [2], the authors showed that f_p is operator monotone for $1 \leq p \leq 2$. In the same category, we observe that the function

$$g_p(t) := \frac{t-1}{t^p - 1}$$

is an operator monotone function for $p \in (0, 1]$, see [3].

It is well known that the logarithmic function ln is operator monotone and in [3], the author proved that the functions

$$f(t) = t(1+t) \ln\left(1 + \frac{1}{t}\right), \quad g(t) = \frac{1}{(1+t) \ln\left(1 + \frac{1}{t}\right)}$$

on $(0, \infty)$ are also operator monotone.

Let A and B be strictly positive operators on a Hilbert space H such that $B - A \geq m1_H > 0$. In 2015, T. Furuta [4] obtained the following result for any non-constant operator monotone function f on $[0, \infty)$:

$$f(B) - f(A) \geq f(\|A\| + m) - f(\|A\|) \geq f(\|B\|) - f(\|B\| - m) > 0. \tag{1.2}$$

If $B > A > 0$, then

$$
\begin{aligned}
f(B) - f(A) &\geq f\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right) - f(\|A\|) \\
&\geq f(\|B\|) - f\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right) > 0.
\end{aligned} \tag{1.3}
$$

The inequality between the first and third term in (1.3) was obtained earlier by H. Zuo and G. Duan in [9].

By taking $f(t) = t^r$, $r \in (0,1]$ in (1.3), Furuta obtained the following refinement of the celebrated Löwner–Heinz inequality

$$
\begin{aligned}
B^r - A^r &\geq \left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r - \|A\|^r \\
&\geq \|B\|^r - \left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r > 0
\end{aligned} \tag{1.4}
$$

provided $B > A > 0$.

With the same assumptions for A and B, we have the logarithmic inequality [4]:

$$
\begin{aligned}
\ln B - \ln A &\geq \ln\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right) - \ln(\|A\|) \\
&\geq \ln(\|B\|) - \ln\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right) > 0.
\end{aligned} \tag{1.5}
$$

Notice that the inequalities between the first and third terms in (1.4) and (1.5) were obtained earlier by M. S. Moslehian and H. Najafi in [8].

Motivated by the above results, we show in this paper that if $f : [0,\infty) \to \mathbb{R}$ is operator monotone on $[0,\infty)$ and there exist positive numbers $d > c > 0$ such that the condition $d1_H \geq B - A \geq c1_H > 0$ is satisfied, then

$$
d\,\frac{f(c) - f(0)}{c} 1_H \geq f(B) - f(A) \geq c\,\frac{f(d + \|A\|) - f(\|A\|)}{d} 1_H \geq 0. \tag{1.6}
$$

Some examples of interest, including a refinement and a reverse of the Löwner–Heinz inequality, are also provided.

2. Main Results.

We have:

Theorem 2. *Assume that $f : [0,\infty) \to \mathbb{R}$ is operator monotone on $[0,\infty)$ given by representation (1.1). Let $A \geq 0$ and assume that there exist positive numbers $d > c > 0$ such that*

$$
d1_H \geq B - A \geq c1_H > 0. \tag{2.1}
$$

Then

$$d\left(\frac{f(c)-f(0)}{c}-b\right)1_H \geq f(B)-f(A)-b(B-A) \geq c\left(\frac{f(d+\|A\|)-f(\|A\|)}{d}-b\right)1_H \geq 0. \tag{2.2}$$

Proof. Since the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$, then f can be written as in the equation (1.1) and for A, $B\geq 0$ we have the representation

$$f(B)-f(A) = b(B-A)+\int_0^\infty s\left[B(B+s1_H)^{-1}-A(A+s1_H)^{-1}\right]dm(s). \tag{2.3}$$

Observe that for $s>0$,

$$\begin{aligned}
&B(B+s1_H)^{-1}-A(A+s1_H)^{-1}\\
&=(B+s1_H-s1_H)(B+s1_H)^{-1}-(A+s1_H-s1_H)(A+s1_H)^{-1}\\
&=(B+s1_H)(B+s1_H)^{-1}-s1_H(B+s1_H)^{-1}\\
&\quad-(A+s1_H)(A+s1_H)^{-1}+s1_H(A+s1_H)^{-1}\\
&=1_H-s1_H(B+s1_H)^{-1}-1_H+s1_H(A+s1_H)^{-1}\\
&=s\left[(A+s1_H)^{-1}-(B+s1_H)^{-1}\right].
\end{aligned}$$

Therefore, (2.3) becomes (see also [4])

$$f(B)-f(A) = b(B-A)+\int_0^\infty s^2\left[(A+s1_H)^{-1}-(B+s1_H)^{-1}\right]dm(s). \tag{2.4}$$

The function $g(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla g_T(S):=\lim_{t\to 0}\left[\frac{g(T+tS)-g(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.5}$$

for T, $S>0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for selfadjoint operators C, D with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$g_{C,D}(t)=g((1-t)C+tD),\ t\in[0,1].$$

If $g_{C,D}$ is Gâteaux differentiable on the segment

$$[C,D]:=\{(1-t)C+tD,\ t\in[0,1]\},$$

then, by the properties of the Bochner integral, we have

$$g(D) - g(C) = \int_0^1 \frac{d}{dt}(g_{C,D}(t))\,dt = \int_0^1 \nabla g_{(1-t)C+tD}(D-C)\,dt. \tag{2.6}$$

If we write this equality for the function $g(t) = -t^{-1}$ and $C,\ D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.7}$$

Now, if we replace in (2.7): $C = A + s1_H$ and $D = B + s1_H$ for $s > 0$, then we get

$$\begin{aligned} &(A+s1_H)^{-1} - (B+s1_H)^{-1} \\ &= \int_0^1 ((1-t)A + tB + s1_H)^{-1}(B-A)((1-t)A+tB+s1_H)^{-1}\,dt. \end{aligned} \tag{2.8}$$

By the representation (2.4), we derive the following identity of interest

$$\begin{aligned} f(B) - f(A) &= b(B-A) \\ &\quad + \int_0^\infty s^2\left[\int_0^1 ((1-t)A+tB+s1_H)^{-1}\right. \\ &\qquad \left.\times (B-A)((1-t)A+tB+s1_H)^{-1}\,dt\right] dm(s) \end{aligned} \tag{2.9}$$

for $A,\ B \geq 0$.

From the representation (2.9) we get

$$f(x) - f(0) - bx = \int_0^\infty s^2\left(\int_0^1 (tx+s)^{-1}x(tx+s)^{-1}\,dt\right) dm(s)$$

for $B = x1_H$, $A = 0$, which for $x > 0$ gives

$$\frac{f(x) - f(0)}{x} - b = \int_0^\infty s^2\left(\int_0^1 (tx+s)^{-2}\,dt\right) dm(s). \tag{2.10}$$

Since $0 < c1_H \leq B - A \leq d1_H$, we have

$$\begin{aligned} &c((1-t)A + tB + s1_H)^{-2} \\ &\quad \leq ((1-t)A+tB+s1_H)^{-1}(B-A)((1-t)A+tB+s1_H)^{-1} \\ &\quad \leq d((1-t)A+tB+s1_H)^{-2} \end{aligned}$$

for $t \in [0,1]$, $s > 0$ and by (2.9), we get

$$\begin{aligned} &c\int_0^\infty s^2\left(\int_0^1 ((1-t)A+tB+s1_H)^{-2}\,dt\right) dm(s) \\ &\quad \leq f(B) - f(A) - b(B-A) \\ &\quad \leq d\int_0^\infty s^2\left(\int_0^1 ((1-t)A+tB+s1_H)^{-2}\,dt\right) dm(s). \end{aligned} \tag{2.11}$$

Observe that for $t \in [0,1]$ and $s > 0$ we have

$$\begin{aligned}(1-t)A + tB + s1_H &= A + t(B-A) + s1_H \\ &\geq 0 + tc1_H + s1_H = (tc+s)1_H.\end{aligned}$$

This implies that

$$((1-t)A + tB + s1_H)^{-1} \leq (tc+s)^{-1} 1_H.$$

Therefore

$$\begin{aligned}&\int_0^\infty s^2 \left(\int_0^1 ((1-t)A + tB + s1_H)^{-2} dt \right) dm(s) \\ &\leq \int_0^\infty s^2 \left(\int_0^1 (tc+s)^{-2} dt \right) dm(s) 1_H \\ &= \left(\frac{f(c) - f(0)}{c} - b \right) 1_H \text{ (by (2.10))}\end{aligned}$$

and by (2.11), we get

$$f(B) - f(A) - b(B-A) \leq d \left(\frac{f(c) - f(0)}{c} - b \right) 1_H. \tag{2.12}$$

We also have

$$\begin{aligned}(1-t)A + tB + s1_H &= A + t(B-A) + s1_H \leq A + td1_H + s1_H \\ &= (1-t)A + t(d1_H + A) + s1_H.\end{aligned}$$

Since $A \leq \|A\| 1_H$, then

$$(1-t)A + t(d1_H + A) + s1_H \leq ((1-t)\|A\| + t(d + \|A\|) + s) 1_H,$$

which implies that

$$(1-t)A + tB + s1_H \leq ((1-t)\|A\| + t(d + \|A\|) + s) 1_H$$

for $t \in [0,1]$ and $s > 0$.

This implies that

$$((1-t)A + tB + s1_H)^{-1} \geq ((1-t)\|A\| + t(d + \|A\|) + s)^{-1} 1_H$$

and

$$((1-t)A + tB + s1_H)^{-2} \geq ((1-t)\|A\| + t(d + \|A\|) + s)^{-2} 1_H$$

for $t \in [0,1]$ and $s > 0$.

Therefore

$$\begin{aligned}&\int_0^\infty s^2 \left(\int_0^1 ((1-t)A + tB + s1_H)^{-2} dt \right) dm(s) \\ &\geq \int_0^\infty s^2 \left(\int_0^1 ((1-t)\|A\| + t(d + \|A\|) + s)^{-2} dt \right) dm(s) 1_H \; (\geq 0)\end{aligned}$$

$$= \frac{1}{d} \int_0^\infty s^2 \left(\int_0^1 ((1-t)\|A\| + t(d+\|A\|) + s)^{-1} (d + \|A\| - \|A\|) \right.$$
$$\left. \times ((1-t)\|A\| + t(d+\|A\|) + s)^{-1} dt \right) dm(s) 1_H$$
$$= \frac{1}{d} [(f(d+\|A\|) - f(\|A\|) - bd)] 1_H$$
(by identity (2.9) for $d + \|A\|$ and $\|A\|$)
$$= \left(\frac{f(d+\|A\|) - f(\|A\|)}{d} - b \right) 1_H \geq 0.$$

By (2.11), we get

$$\begin{aligned} & f(B) - f(A) - b(B-A) \\ & \geq c \int_0^\infty s^2 \left(\int_0^1 ((1-t)A + tB + s1_H)^{-2} dt \right) dm(s) \\ & \geq c \left(\frac{f(d+\|A\|) - f(\|A\|)}{d} - b \right) 1_H \geq 0. \end{aligned} \tag{2.13}$$

The inequalities (2.12) and (2.13) imply (2.2). □

From the first inequality in (2.2) we get

$$d \frac{f(c) - f(0)}{c} 1_H - b[d1_H - (B-A)] \geq f(B) - f(A)$$

and since $d1_H - (B-A) \geq 0$ and $b \geq 0$,

$$d \frac{f(c) - f(0)}{c} 1_H \geq d \frac{f(c) - f(0)}{c} 1_H - b[d1_H - (B-A)].$$

From the second inequality in (2.2) we have

$$\begin{aligned} f(B) - f(A) & \geq b[(B-A) - c] + c \frac{f(d+\|A\|) - f(\|A\|)}{d} 1_H \\ & \geq c \frac{f(d+\|A\|) - f(\|A\|)}{d} 1_H \geq 0 \end{aligned}$$

since $b[(B-A) - c1_H] \geq 0$.

Therefore we have the following result which does not contain the value b:

Corollary 1. *Assume that* $f : [0,\infty) \to \mathbb{R}$ *is operator monotone on* $[0,\infty)$, $A \geq 0$ *and that there exist positive numbers* $d > c > 0$ *such that the condition* (2.1) *is satisfied. Then*

$$d \frac{f(c) - f(0)}{c} 1_H \geq f(B) - f(A) \geq c \frac{f(d+\|A\|) - f(\|A\|)}{d} 1_H \geq 0. \tag{2.14}$$

Remark 1. If we take $f(t) = t^r$, $r \in (0,1]$, in (2.14), then we get

$$dc^{r-1} 1_H \geq B^r - A^r \geq c \frac{(d+\|A\|)^r - \|A\|^r}{d} 1_H \geq 0, \tag{2.15}$$

provided that the condition (2.1) is satisfied and $A \geq 0$.

Let $\varepsilon > 0$. Consider the function $f_\varepsilon : [0,\infty) \to \mathbb{R}$, $f_\varepsilon(t) = \ln(\varepsilon + t)$. This function is operator monotone on $[0,\infty)$ and by the second inequality in (2.14), we get

$$\begin{aligned}\ln(B+\varepsilon 1_H) - \ln(A+\varepsilon 1_H) \\ \geq c\frac{\ln(d+\|A\|+\varepsilon) - \ln(\|A\|+\varepsilon)}{d}1_H > 0.\end{aligned} \tag{2.16}$$

By taking the limit over $\varepsilon \to 0+$ in (2.16), we get

$$\ln(B) - \ln(A) \geq c\frac{\ln(d+\|A\|) - \ln(\|A\|)}{d}1_H > 0 \tag{2.17}$$

for $d1_H \geq B - A \geq c1_H > 0$ and $A > 0$.

It is well known that if $P \geq 0$, then

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle$$

for all x, $y \in H$.

Therefore, if $T > 0$, then

$$\begin{aligned}0 &\leq \langle x, x\rangle^2 = \left\langle T^{-1}Tx, x\right\rangle^2 = \left\langle Tx, T^{-1}x\right\rangle^2 \\ &\leq \langle Tx, x\rangle \left\langle TT^{-1}x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle\end{aligned}$$

for all $x \in H$.

If $x \in H$, $\|x\| = 1$, then

$$1 \leq \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle \leq \langle Tx, x\rangle \sup_{\|x\|=1} \left\langle x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\frac{1}{\|T^{-1}\|}1_H \leq T. \tag{2.18}$$

Corollary 2. *Assume that $f : [0,\infty) \to \mathbb{R}$ is operator monotone on $[0,\infty)$ and $B > A \geq 0$, then*

$$\begin{aligned}&\|B-A\| \left\|(B-A)^{-1}\right\| \left[f\left(\left\|(B-A)^{-1}\right\|^{-1}\right) - f(0)\right]1_H \\ &\geq f(B) - f(A) \\ &\geq \frac{f(\|B-A\|+\|A\|) - f(\|A\|)}{\left\|(B-A)^{-1}\right\|\|B-A\|}1_H \\ &\geq \frac{f(\|B\|) - f(\|A\|)}{\left\|(B-A)^{-1}\right\|\|B-A\|}1_H \geq 0.\end{aligned} \tag{2.19}$$

Proof. Since $B - A > 0$, by (2.18) we get

$$\frac{1}{\left\|(B-A)^{-1}\right\|}1_H \leq B - A \leq \|B-A\| 1_H.$$

So, if we write the inequality (2.14) for $c = \frac{1}{\|(B-A)^{-1}\|}$ and $d = \|B - A\|$, then we get

$$\begin{aligned} & \|B-A\| \left\|(B-A)^{-1}\right\| \left[f\left(\left\|(B-A)^{-1}\right\|^{-1}\right) - f(0)\,1_H\right] \\ (2.20) \qquad & \geq f(B) - f(A) \\ & \geq \frac{f(\|B-A\| + \|A\|) - f(\|A\|)}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H \geq 0. \end{aligned}$$

Also, we have $\|B - A\| + \|A\| \geq \|B\|$ and since f is nondecreasing, then

$$(2.21) \qquad f(\|B-A\| + \|A\|) - f(\|A\|) \geq f(\|B\|) - f(\|A\|) \geq 0.$$

By (2.20) and (2.21) we derive (2.19). □

Remark 2. By making use of a similar argument as in Remark 1, we can also derive the logarithmic inequality

$$\begin{aligned} \ln(B) - \ln(A) &\geq \frac{\ln(\|B-A\| + \|A\|) - \ln(\|A\|)}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H \\ &\geq \frac{\ln(\|B\|) - \ln(\|A\|)}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H > 0 \end{aligned}$$

for $A > 0$ and $B - A > 0$.

3. Some Examples. Assume that $B > A \geq 0$ and $r \in (0, 1]$. Then by (2.19) we have, for the operator monotone function $f(t) = t^r$ on $[0, \infty)$, the following refinement and reverse of Löwner–Heinz inequality

$$\begin{aligned} & \|B-A\| \left\|(B-A)^{-1}\right\|^{1-r} 1_H \geq B^r - A^r \\ (3.1) \qquad & \geq \frac{(\|B-A\| + \|A\|)^r - \|A\|^r}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H \\ & \geq \frac{\|B\|^r - \|A\|^r}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H > 0. \end{aligned}$$

Consider the function

$$f_0(t) := \begin{cases} \frac{t}{1-t} \ln t & \text{for } t > 0, \\ 0 & \text{for } t = 0, \end{cases}$$

which is operator monotone on $[0, \infty)$. By (2.19), we then have

$$\begin{aligned} & \frac{\|B-A\|}{\left\|(B-A)^{-1}\right\|^{-1} - 1} \ln \left\|(B-A)^{-1}\right\| 1_H \\ (3.2) \qquad & \geq B(1_H - B)^{-1} \ln B - A(1_H - A)^{-1} \ln A \\ & \geq \frac{\frac{\|B\|}{1-\|B\|} \ln \|B\| - \frac{\|A\|}{1-\|A\|} \ln \|A\|}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H > 0 \end{aligned}$$

for $B > A > 0$ and $\|A\|$, $\|B\|$, $\left\|(B-A)^{-1}\right\| \neq 1$.

The function $f(t) = \ln(t+1)$ is also operator monotone on $[0,\infty)$, so by (2.19) we have

$$
\begin{aligned}
&\|B-A\| \left\|(B-A)^{-1}\right\| \ln\left(\left\|(B-A)^{-1}\right\|^{-1} + 1\right) 1_H \\
&\quad \geq \ln(B + 1_H) - \ln(A + 1_H) \\
&\quad \geq \frac{\ln(\|B-A\| + \|A\| + 1) - \ln(\|A\| + 1)}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H \qquad (3.3) \\
&\quad \geq \frac{\ln(\|B\| + 1) - \ln(\|A\| + 1)}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H > 0
\end{aligned}
$$

for $B > A \geq 0$.

Consider the function $f_{-1}(t) = \frac{2t}{1+t}$, $t \in [0,\infty)$, which is operator monotone, then by (2.19) we derive

$$
\begin{aligned}
&\frac{\|B-A\|}{1 + \left\|(B-A)^{-1}\right\|^{-1}} 1_H \\
&\quad \geq B(1_H + B)^{-1} - A(1_H + A)^{-1} \qquad (3.4) \\
&\quad \geq \frac{\|B\| - \|A\|}{\left\|(B-A)^{-1}\right\| \|B-A\| (1 + \|B\|)(1 + \|A\|)} 1_H > 0
\end{aligned}
$$

for $B > A \geq 0$.

The interested reader may state other similar inequalities by employing the operator monotone functions presented in Introduction. We omit the details.

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