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Cullis–Radić determinant of a rectangular matrix which has a number of identical columns

Abstract. In this paper we present how identical columns affect the Cullis–Radić determinant of an $m \times n$ matrix, where $m \leq n$.

1. Introduction. In 1913, Cullis [4] introduced the concept of *determinoid* of a rectangular matrix and it is probably the first published generalization of the determinant of a square matrix. Cullis reserved the name *determinant* for square matrices only, but nowadays this term is also used for rectangular matrices.

Since the original Cullis's definition of the determinant (*determinoid*) of a rectangular matrix is descriptive and requires some nonstandard terminology to be introduced first, we reformulate the idea of Cullis in the following way:

Definition 1.1 (Cullis [4, §3], reformulated). Let $A = \{a_{ij}\}$ be an $m \times n$ matrix with m rows and n columns, with elements a_{ij}, where $1 \leq i \leq m$, $1 \leq j \leq n$, and let $k = \min\{m, n\}$. The determinant of A is defined as the sum

$$\det A = \sum_{P} (-1)^{\alpha(P)} P,$$

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over all products $P = a_{i_1 j_1} \dots a_{i_k j_k}$ of elements of A taken from different rows and different columns with

$$\alpha(P) = \sum_{p=1}^{k} \left[\Big(i_p - \sum_{\substack{1 \le q \le p \\ i_q \le i_p}} 1 \Big) + \Big(j_p - \sum_{\substack{1 \le q \le p \\ j_q \le j_p}} 1 \Big) \right]$$

where the expression in square brackets is equal to the sum of horizontal and vertical steps one must take to pass from the element $a_{i_p j_p}$ to the element a_{11} in the matrix obtained from A by removing rows and columns containing the elements $a_{i_q j_q}$ for $q < p$.

Independently of Cullis, in 1966, Radić [11] proposed the following definition of the determinant, which turns out to be equivalent to the Cullis's definition (the equivalence follows from [4, §30]).

Definition 1.2 (Radić [11, Definition 1]). Let $A = [A_1, \dots, A_n]$ be an $m \times n$ matrix with n columns $A_1, \dots, A_n$ and $m \le n$. The determinant of A is defined as the sum

$$\det A = \sum_{1 \le j_1 < \dots < j_m \le n} (-1)^{r + j_1 + \dots + j_m} \det[A_{j_1}, \dots, A_{j_m}], \tag{1.1}$$

where $r = 1 + 2 + \dots + m$. If $m > n$, then $\det A = \det A^T$.

It is worth noting here that there are also other definitions of the determinant of a rectangular matrix which are not equivalent to the definitions of Cullis and Radić, see for example [2, 3, 5, 9, 18, 19, 21].

The Cullis–Radić determinant of an $m \times n$ matrix, where $m \le n$, and the classical determinant of a square matrix have several common properties, see [4, §5, §27, §32] and [11, 16], for example:

(1) The Cullis–Radić determinant of a matrix is a linear function of its rows.
(2) If a matrix A has two identical rows or one of its rows is a linear combination of other rows, then the Cullis–Radić determinant of A is equal to zero.
(3) Interchanging any two rows of a matrix changes the sign of its Cullis–Radić determinant.
(4) Adding a linear combination of rows to another row does not change the Cullis–Radić determinant.
(5) The Cullis–Radić determinant can be calculated using the Laplace expansion with respect to a row.

More algebraic properties, which characterize the Cullis–Radić determinant, can be found in [1, 4, 6, 8, 12, 13, 14, 16, 20], and some geometric interpretations are presented in [7, 10, 12, 13, 14, 15, 17, 20].

In this paper, we are going to present how identical columns affect the Cullis–Radić determinant of an $m \times n$ matrix, where $m \le n$. The rest of the paper is organized as follows. In Section 2, we consider the Cullis–Radić determinant of matrices with two identical columns. In Section 3 and Section 4, we present the results for matrices which have an arbitrary number of identical adjacent columns and an arbitrary number of identical adjacent pairs of columns. Matrices formed by identical adjacent sequences of columns are the subject of Section 5.

2. Two identical columns. First, we introduce some useful notation and recall from [6] how interchanging two columns affects the determinant of a rectangular matrix.

Let $A = [A_1, \ldots, A_n]$ be an $m \times n$ matrix, where $m \le n$. We use the following notation:

(i) $A_{i\leftrightarrow j}$, where $i, j \in \{1, 2, \ldots, n\}$, $i \ne j$, denotes the matrix obtained from A by interchanging columns A_i and A_j,

(ii) $D_i(A) = [A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n]$, where $i \in \{1, 2, \ldots, n\}$ and $n > 1$, denotes the matrix obtained from A by removing the column A_i,

(iii) $I_i^K(A) = [A_1, \ldots, A_{i-1}, K, A_i, A_{i+1}, \ldots, A_n]$, where $i \in \{1, 2, \ldots, n\}$, denotes the matrix obtained from A by inserting the column K before the i-th column of A.

Lemma 2.1 ([6, Theorem 2.6]). *Let A be an $m \times (m+1)$ matrix. Then for all $i, j \in \{1, 2, \ldots, m+1\}$ such that $i \ne j$, we have*

$$\det A + \det A_{i\leftrightarrow j} = 0.$$

Lemma 2.2 ([6, Theorem 2.7]). *Let $A = [A_1, \ldots, A_n]$ be an $m \times n$ matrix, where $m \le n$. Then*

$$\begin{aligned}\det A + \det A_{i\leftrightarrow j} = {} & 2 \sum_{\substack{1\le j_1<\ldots<j_m\le n\\ i,j\notin\{j_1,\ldots,j_m\}}} (-1)^{r+j_1+j_2+\ldots+j_m} \det[A_{j_1}, A_{j_2}, \ldots, A_{j_m}] \\ & + 2 \sum_{\substack{1\le j_1<\ldots<j_m\le n\\ (i\in J,\, j\notin J \text{ or } i\notin J,\, j\in J)\\ J=\{\min\{i,j\},\ldots,\max\{i,j\}\}\setminus\{j_1,\ldots,j_m\}\\ \operatorname{card}(J)\equiv 0 \pmod 2}} (-1)^{r+j_1+j_2+\ldots+j_m} \det[A_{j_1}, A_{j_2}, \ldots, A_{j_m}],\end{aligned}$$

where $r = 1 + 2 + \ldots + m$ and $\operatorname{card}(X)$ *stands for the cardinality of X.*

Corollary 2.3 ([6, Corollary 2.8]). *Let A be an $m \times n$ matrix, where $m \le n$. Then for every $i \in \{1, 2, \ldots, n-1\}$, we have*

$$\det A + \det A_{i\leftrightarrow(i+1)} = \begin{cases} 0, & \text{if } n \in \{m, m+1\}, \\ 2 \det D_i(D_{i+1}(A)), & \text{if } n \ge m+2. \end{cases}$$

The following theorem gives a sufficient condition for a matrix to have the determinant equal to zero and shows that the determinant of a matrix which has two identical columns can be expressed by determinants of matrices obtained from the given matrix by removing one or both identical columns.

Theorem 2.4. *Let* $A = [A_1, \ldots, A_n]$ *be an* $m \times n$ *matrix, where* $m \leq n$ *and* $A_i = A_j$ *for some* $i, j \in \{1, 2, \ldots, n\}$ *such that* $i < j$. *Then*

$$\det A = \begin{cases} 0, & \text{if } n \in \{m, m+1\}, \\ \det D_i(D_j(A)), & \text{if } n \geq m+2 \text{ and } j = i+1, \\ 2 \sum_{p=i+1}^{j-1} (-1)^{p-i-1} \det D_i(D_p(A)) & \\ \quad +(-1)^{j-i-1} \det D_i(D_j(A)), & \text{if } n \geq m+2 \text{ and } j > i+1. \end{cases}$$

Proof. If $n = m+1$, then we apply Lemma 2.1 and obtain

$$2 \det A = \det A + \det A_{i \leftrightarrow j} = 0.$$

In the case where $n \geq m+2$ and $j = i+1$, by Corollary 2.3, we have

$$2 \det A = \det A + \det A_{i \leftrightarrow j} = 2 \det D_i(D_j(A)).$$

If $n \geq m+2$ and $j > i+1$, then for every $k \in \{0, 1, \ldots, j-i-2\}$ Corollary 2.3 yields

$$\det I_{i+k}^{A_i}(D_i(A)) = 2 \det D_i(D_{i+k+1}(A)) - \det I_{i+k+1}^{A_i}(D_i(A)). \tag{2.1}$$

From the previous case it follows that

$$\det I_{j-1}^{A_i}(D_i(A)) = \det D_i(D_j(A)).$$

Therefore, using (2.1), we obtain

$$\begin{aligned} \det A &= \det I_i^{A_i}(D_i(A)) = 2 \det D_i(D_{i+1}(A)) - \det I_{i+1}^{A_i}(D_i(A)) \\ &= 2 \det D_i(D_{i+1}(A)) - 2 \det D_i(D_{i+2}(A)) + \det I_{i+2}^{A_i}(D_i(A)) \\ &= 2 \sum_{p=i+1}^{j-1} (-1)^{p-i-1} \det D_i(D_p(A)) + (-1)^{j-i-1} \det D_i(D_j(A)). \quad \square \end{aligned}$$

Remark 2.5. In the proof of Theorem 2.4, instead of (2.1) the following relation:

$$\det I_{j-k}^{A_j}(D_j(A)) = 2 \det D_j(D_{j-k-1}(A)) - \det I_{j-k-1}^{A_j}(D_j(A))$$

for $k \in \{0, 1, \ldots, j-i-2\}$ can also be used, yielding

$$\det A = 2 \sum_{p=i+1}^{j-1} (-1)^{j-p-1} \det D_j(D_p(A)) + (-1)^{j-i-1} \det D_j(D_i(A))$$

for $n \geq m+2$ and $j > i+1$.

Example 2.6. Let $A = [A_1, A_2, A_3, A_4, A_5, A_6]$ be a 3×6 matrix.
(a) If $A_2 = A_5$, then

$$\begin{aligned}\det A &= 2\det D_2(D_3(A)) - 2\det D_2(D_4(A)) + \det D_2(D_5(A))\\ &= 2\det[A_1, A_4, A_5, A_6] - 2\det[A_1, A_3, A_5, A_6] + \det[A_1, A_3, A_4, A_6]\end{aligned}$$

and

$$\begin{aligned}\det A &= 2\det D_5(D_4(A)) - 2\det D_5(D_3(A)) + \det D_5(D_2(A))\\ &= 2\det[A_1, A_2, A_3, A_6] - 2\det[A_1, A_2, A_4, A_6] + \det[A_1, A_3, A_4, A_6].\end{aligned}$$

(b) If $A_2 = A_3$, then

$$\det A = \det[A_1, A_4, A_5, A_6].$$

3. An arbitrary number of identical adjacent columns. In this section, we present two theorems which follow easily from the fact that, while calculating the determinant of an $m \times n$ matrix, two adjacent identical columns can be canceled if $n \geq m + 2$ (see Theorem 2.4).

Theorem 3.1. *Let $A = [A_1, \ldots, A_n]$ be an $m \times n$ matrix, where $m \leq n$. Fix $i \in \{1, 2, \ldots, n\}$ and replace the column A_i in the matrix A with k copies of A_i, where $k \geq 2$, obtaining in this way a matrix B of the form*

$$B = [A_1, \ldots, A_{i-1}, \underbrace{A_i, \ldots, A_i}_{k \text{ columns}}, A_{i+1}, \ldots, A_n].$$

Then

$$\det B = \begin{cases} \det A, & \text{if } k \text{ is odd},\\ 0, & \text{if } k \text{ is even and } m = n,\\ \det D_i(A), & \text{if } k \text{ is even and } m < n.\end{cases}$$

Theorem 3.2. *Let $A = [A_1, \ldots, A_n]$ be an $m \times n$ matrix, where $m \leq n$. For each $i \in \{1, 2, \ldots, n\}$ replace the column A_i in the matrix A with k_i copies of A_i, where $k_i \geq 1$, obtaining in this way a matrix B of the form*

$$B = [\underbrace{A_1, \ldots, A_1}_{k_1 \text{ columns}}, \underbrace{A_2, \ldots, A_2}_{k_2 \text{ columns}}, \ldots, \underbrace{A_n, \ldots, A_n}_{k_n \text{ columns}}].$$

If p is the number of all odd integers among $k_1, \ldots, k_n$ and $(k_{j_1}, \ldots, k_{j_p})$ is the subsequence of all odd numbers taken from the sequence $(k_1, \ldots, k_n)$, then

$$\det B = \begin{cases} \det A, & \text{if } p = n \text{ (all } k_i\text{'s are odd)},\\ \det[A_{j_1}, \ldots, A_{j_p}], & \text{if } m \leq p < n,\\ 0, & \text{if } p < m.\end{cases}$$

Example 3.3. If $A = [A_1, A_2, A_2, A_2, A_3, A_4]$ is a 2×6 matrix and

$$B = [B_1, B_2, B_2, B_3, B_3, B_3, B_4, B_5, B_5, B_5, B_5, B_6]$$

is a 3×12 matrix, then

$$\det A = \det[A_1, A_2, A_3, A_4] \text{ and } \det B = \det[B_1, B_3, B_4, B_6].$$

4. An arbitrary number of identical adjacent pairs of columns.

Theorem 4.1. *Let $A = [A_1, \dots, A_n]$ be an $m \times n$ matrix, where $m \le n$. Fix $i \in \{1, 2, \dots, m-1\}$ and replace the pair of columns A_i, A_{i+1} in the matrix A with k copies of the pair A_i, A_{i+1}, where $k \ge 2$, obtaining in this way a matrix B_k of the form*

$$B_k = [A_1, \dots, A_{i-1}, \underbrace{A_i, A_{i+1}, \dots, A_i, A_{i+1}}_{k \text{ pairs } A_i A_{i+1}}, A_{i+2}, \dots, A_n].$$

Then

$$\det B_k = \begin{cases} k \det A, & \text{if } n \in \{m, m+1\}, \\ k \det A - (k-1) \det D_i(D_{i+1}(A)), & \text{if } n \ge m+2. \end{cases} \tag{4.1}$$

Proof. First, we prove by induction that for every positive integer k

$$\det B_k = (k-1) \det B_2 - (k-2) \det A. \tag{4.2}$$

The cases $k = 1$ and $k = 2$ are easy. For $k \ge 3$, applying Corollary 2.3, we have

$$\begin{aligned} \det B_k + \det[A_1, \dots, A_{i-1}, A_{i+1}, A_i, \underbrace{A_i, A_{i+1}, \dots, A_i, A_{i+1}}_{k-1 \text{ pairs } A_i A_{i+1}}, A_{i+2}, \dots, A_n] \\ = 2 \det B_{k-1}. \end{aligned} \tag{4.3}$$

Theorem 3.1 yields

$$\det B_k + \det B_{k-2} = 2 \det B_{k-1}.$$

Assuming that

$$\det B_{k-1} = (k-2) \det B_2 - (k-3) \det A,$$
$$\det B_{k-2} = (k-3) \det B_2 - (k-4) \det A,$$

we obtain

$$\det B_k = 2 \det B_{k-1} - \det B_{k-2} = (k-1) \det B_2 - (k-2) \det A.$$

Considering (4.3) for $k = 2$, and then applying Theorem 2.4, we have

$$\det B_2 = \begin{cases} 2 \det A, & \text{if } n \in \{m, m+1\}, \\ 2 \det A - \det D_i(D_{i+1}(A)), & \text{if } n \ge m+2. \end{cases} \tag{4.4}$$

Now, combining (4.2) and (4.4), we obtain (4.1). □

Example 4.2. Let

$$A = [A_1, A_2, A_3, A_4] \text{ and } B = [A_1, A_2, A_1, A_2, A_1, A_2, A_3, A_4].$$

(a) If A is a 2×4 matrix and B is a 2×8 matrix, then

$$\det B = 3 \det A - 2 \det[A_3, A_4].$$

(b) If A is a 3×4 matrix and B is a 3×8 matrix, then

$$\det B = 3 \det A.$$

5. An arbitrary number of identical adjacent sequences of columns which form the entire matrix. In this section we prove the following theorem.

Theorem 5.1. *Let k be a positive integer. If A is an $m \times n$ matrix, where $m \leq n$, then*

$$\det[\underbrace{A\,, A\,, \ldots, A}_{k \text{ copies of } A}] = \begin{cases} k^{m/2} \det A, & \text{if } m \text{ is even}, \\ k^{(m+1)/2} \det A, & \text{if } m \text{ is odd and } n \text{ is even}, \\ 0, & \text{if } m, n \text{ are odd and } k \text{ is even}, \\ k^{(m-1)/2} \det A, & \text{if } m, n, k \text{ are odd.} \end{cases}$$

Proof. The proof follows immediately from Lemma 5.2, Lemma 5.3 and Lemma 5.7, which are proved below. □

Before we start with the first lemma, we introduce some useful notation. Assume that

$$A = [A_1, \ldots, A_n]$$

is an $m \times n$ matrix with n columns $A_1, \ldots, A_n$, $m \leq n$, and

$$B = [\underbrace{A\,, A\,, \ldots, A}_{k \text{ copies of } A}] = [\mathcal{B}_1, \ldots, \mathcal{B}_k] = [B_1, \ldots, B_{nk}]$$

is an $m \times nk$ matrix, where $\mathcal{B}_i = A$ for every $i = 1, 2, \ldots, k$, and $B_1, \ldots, B_{nk}$ are the columns of B.

For every sequence $a = (a_1, \ldots, a_k)$ of nonnegative integers such that $\sum_{i=1}^{k} a_i = m$, define $G_a(A)$ to be the set of $m \times m$ matrices which have a_1 columns taken from the matrix $\mathcal{B}_1$, followed by a_2 columns taken from the matrix $\mathcal{B}_2$, etc., followed by a_k columns taken from the matrix $\mathcal{B}_k$ — in each group of a_i columns, the columns are arranged in the same order as they were arranged in the matrix $\mathcal{B}_i$, $i = 1, 2, \ldots, n$.

Moreover, for every matrix $M = [M_1, \ldots, M_m] \in G_a(A)$ such that $M_i = B_{j_i}$ for each $i = 1, 2, \ldots, m$, define

$$c(a, M) = \sum_{i=1}^{m} j_i.$$

Lemma 5.2. *Let $A = [A_1, \ldots, A_n]$ be an $m \times n$ matrix with n columns $A_1, \ldots, A_n$, where $m \leq n$, and let $B = [\mathcal{B}_1, \ldots, \mathcal{B}_k]$ be an $m \times nk$ matrix, where $\mathcal{B}_i = A$ for each $i = 1, 2, \ldots, k$. Then*

$$\det B = \sum_{\substack{a=(a_1,\ldots,a_k) \\ a_1+\ldots+a_k=m \\ a_i \in \mathbb{Z},\ a_i \geq 0,\ i=1,2,\ldots,k}} \ \sum_{M \in G_a(A)} (-1)^{r+c(a,M)} \det M,$$

where $r = 1 + 2 + \ldots + m$.

Proof. The proof follows easily from Definition 1.2. □

Let n, m, k be positive integers and $\widetilde{n} = \frac{1-(-1)^n}{2} \in \{0,1\}$. For any sequence $(a_1, \ldots, a_k)$ of nonnegative integers satisfying $\sum_{i=1}^k a_i = m$, define $S^{\widetilde{n}}_{(a_1,\ldots,a_k)}$ by the following formulas:

(5.1a) $$S^{\widetilde{n}}_{(m)} = 1,$$

(5.1b) $$S^{\widetilde{n}}_{(m,0)} = 1,$$

(5.1c) $$S^{\widetilde{n}}_{(0,m)} = (-1)^{mn},$$

(5.1d) $$S^{\widetilde{n}}_{(1,m-1)} = \begin{cases} 0, & \text{if } m \text{ is even,} \\ 1, & \text{if } m \text{ is odd,} \end{cases}$$

(5.1e) $$S^{\widetilde{n}}_{(m-1,1)} = \begin{cases} 0, & \text{if } m \text{ is even,} \\ (-1)^n, & \text{if } m \text{ is odd,} \end{cases}$$

(5.1f) $$S^{\widetilde{n}}_{(u,v)} = \sum_{i=0}^{v} (-1)^{i(u+n)} S^{\widetilde{n}}_{(u-1,v-i)},$$
where $u \geq 2$, $v \geq 2$ are integers and $u + v = m$,

(5.1g) $$S^{\widetilde{n}}_{(a_1,\ldots,a_p,0,\ldots,0)} = S^{\widetilde{n}}_{(a_1,\ldots,a_p)},$$
where $p < k$, $a_1 \neq 0$, $a_p \neq 0$, $k \geq 3$,

(5.1h) $$S^{\widetilde{n}}_{(a_1,\ldots,a_k)} = (-1)^{mnp} S^{\widetilde{n}}_{(a_{p+1},\ldots,a_k)},$$
where $p < k$, $a_1 = \ldots = a_p = 0$, $a_{p+1} \neq 0$, $k \geq 3$,

(5.1i) $$S^{\widetilde{n}}_{(a_1,\ldots,a_{k-2},a_{k-1},a_k)} = S^{\widetilde{n}}_{(a_1,\ldots,a_{k-2},a_{k-1}+a_k)} S^{\widetilde{n}}_{(a_{k-1},a_k)},$$
where $a_1 \neq 0$, $a_k \neq 0$, $k \geq 3$.

Lemma 5.3. *Let $A = [A_1, \ldots, A_n]$ be an $m \times n$ matrix with n columns $A_1, \ldots, A_n$, where $m \leq n$, and let $B = [\mathcal{B}_1, \ldots, \mathcal{B}_k]$ be an $m \times nk$ matrix, where $\mathcal{B}_i = A$ for each $i = 1, 2, \ldots, k$. For every sequence of nonnegative integers $a = (a_1, \ldots, a_k)$ such that $\sum_{i=1}^k a_i = m$, we have*

(5.2) $$\sum_{M \in G_a(A)} (-1)^{r+c(a,M)} \det M = S^{\widetilde{n}}_a \det A,$$

where $r = 1 + \ldots + m$ and $\widetilde{n} = \frac{1-(-1)^n}{2}$.

Proof. The proof is done by induction on k and it is divided into nine steps. The first five steps (a)–(e) cover the base cases. The remaining steps (f)–(i) complete the induction.

(a) For $k = 1$, we have $B = A$ and (5.2) follows immediately from Lemma 5.2 and formula (5.1a).

(b) For $k = 2$ and $a = (m, 0)$, we have $G_a(A) = G_{(m,0)}(A) = G_{(m)}(A)$ and $S^{\widetilde{n}}_a = S^{\widetilde{n}}_{(m,0)} = S^{\widetilde{n}}_{(m)}$. Therefore, applying case (a), we obtain (5.2).

(c) For $k = 2$ and $a = (0, m)$, for every matrix $M = [M_1, \dots, M_m] \in G_a(A)$ such that $M_i = B_{j_i}$ for each $i = 1, 2, \dots, m$, we have $M \in G_{(m)}(A)$ and

$$c(a, M) = \sum_{i=1}^{m} j_i = \sum_{i=1}^{m} (j_i - n) + mn = c((m), M) + mn.$$

Since there is a one-to-one correspondence between matrices in $G_a(A)$ and $G_{(m)}(A)$, we have

$$\begin{aligned}\sum_{M \in G_a(A)} (-1)^{r+c(a,M)} \det M &= (-1)^{mn} \sum_{M \in G_{(m)}(A)} (-1)^{r+c((m),M)} \det M \\ &= (-1)^{mn} \det A.\end{aligned}$$

Now, applying (5.1c), we obtain (5.2).

(d) For $k = 2$ and $a = (1, m-1)$, every matrix $M \in G_a(A)$ can be represented in the form $M = [A_{p_1}, \dots, A_{p_m}]$, where $(p_2, \dots, p_m)$ is an increasing sequence of integers and $p_i \in \{1, 2, \dots, n\}$, $i = 1, 2, \dots, m$. Moreover, for every sequence $(p_1, \dots, p_m)$ of integers such that $(p_2, \dots, p_m)$ is increasing and $p_i \in \{1, 2, \dots, n\}$, $i = 1, 2, \dots, k$, the matrix $[A_{p_1}, \dots, A_{p_m}]$ is an element of $G_a(A)$.

Fix a nondecreasing sequence $(p_1, \dots, p_m)$ of integers from $\{1, 2, \dots, n\}$ and consider the sum

$$\mathcal{S}(p_1, \dots, p_m) = \sum (-1)^{r+c(a,M)} \det M$$

over all matrices $M \in G_a(A)$ such that $A_{p_1}, \dots, A_{p_m}$ are the columns of M. Since any of the columns $A_{p_1}, \dots, A_{p_m}$ can be the first column of M, we have

$$\begin{aligned}\mathcal{S}(p_1, \dots, p_m) &= (-1)^{r+p_1+\sum_{i=2}^{m}(p_i+n)} \det[A_{p_1}, A_{p_2}, \dots, A_{p_m}] \\ &\quad + \sum_{j=2}^{m} (-1)^{r+p_j+\sum_{\substack{i=1\\ i\neq j}}^{m}(p_i+n)} \det[A_{p_j}, A_{p_1}, \dots, A_{p_{j-1}}, A_{p_{j+1}}, \dots, A_{p_m}] \\ &= \begin{cases} (-1)^{r+\sum_{i=1}^{m} p_i} \det[A_{p_1}, \dots, A_{p_m}], & \text{if } m \text{ is odd and } p_1 < \dots < p_m, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Therefore

$$\sum_{M \in G_a(A)} (-1)^{r+c(a,M)} \det M = \sum_{p_1 < \dots < p_m} \mathcal{S}(p_1, \dots, p_m) = \begin{cases} \det A, & \text{if } m \text{ is odd}, \\ 0, & \text{if } m \text{ is even}, \end{cases}$$

and now (5.2) follows easily from (5.1d).

(e) For $k = 2$ and $a = (m-1, 1)$, the proof is similar to the previous case. Fix a nondecreasing sequence $(p_1, \ldots, p_m)$ of integers from $\{1, 2, \ldots, n\}$ and consider the sum

$$\mathcal{S}(p_1, \ldots, p_m) = \sum (-1)^{r+c(a,M)} \det M$$

over all matrices $M \in G_a(A)$ such that $A_{p_1}, \ldots, A_{p_m}$ are the columns of M. The calculations

$$\begin{aligned} &\mathcal{S}(p_1, \ldots, p_m) \\ &= \sum_{j=1}^{m-1} (-1)^{r + \sum_{i=1, i\neq j}^{m} p_i + (p_j+n)} \det[A_{p_1}, \ldots, A_{p_{j-1}}, A_{p_{j+1}}, \ldots, A_{p_m}, A_{p_j}] \\ &\qquad + (-1)^{r+\sum_{i=1}^{m-1} p_i + (p_m+n)} \det[A_{p_1}, \ldots, A_{p_{m-1}}, A_{p_m}] \\ &= \begin{cases} (-1)^{r+\sum_{i=1}^{m} p_i + n} \det[A_{p_1}, \ldots, A_{p_m}], & \text{if } m \text{ is odd and } p_1 < \ldots < p_m, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

yield

$$\begin{aligned} \sum_{M \in G_a(A)} (-1)^{r+c(a,M)} \det M &= \sum_{p_1 < \ldots < p_m} \mathcal{S}(p_1, \ldots, p_m) \\ &= \begin{cases} (-1)^n \det A, & \text{if } m \text{ is odd}, \\ 0, & \text{if } m \text{ is even}. \end{cases} \end{aligned}$$

Thus, applying (5.1e), we obtain (5.2).

(f) Let $k = 2$ and $a = (u, v)$, where $u \geq 2$, $v \geq 2$ are integers and $u + v = m$. In this case we prove (5.2) by induction. Assume that for every $\widetilde{m} \times n$ matrix $\widetilde{A}$ and for every $\widetilde{a} = (u-1, \widetilde{v})$, where $\widetilde{v} \geq 2$ is an integer and $u + \widetilde{v} = \widetilde{m} < m$, we have

$$\sum_{M \in G_{\widetilde{a}}(\widetilde{A})} (-1)^{\widetilde{r}+c(\widetilde{a},M)} \det M = S_{\widetilde{a}}^{\widetilde{m}} \det \widetilde{A},$$

where $\widetilde{r} = 1 + 2 + \ldots + \widetilde{m}$.

Notice that

$$\sum_{M \in G_a(A)} (-1)^{r+c(a,M)} \det M = \sum_{1 \leq p_1 < \ldots < p_m \leq n} \left(\sum_{j=1}^{v+1} \mathcal{S}_j(p_1, \ldots, p_m) \right),$$

where

$$\mathcal{S}_j(p_1, \ldots, p_m) = \sum (-1)^{r+c(a,M)} \det M$$

is the sum over all matrices $M \in G_a(A)$ such that A_{p_j} is the first column of M and the columns A_{p_i}, where $i \in \{1, 2, \ldots, m\} \setminus \{j\}$, are the other columns of M. Moreover, for $1 \le p_1 < \ldots < p_m \le n$, we have

$$\mathcal{S}_j(p_1,\ldots,p_m) = \sum_{\substack{1\le r_1<\ldots<r_u\le n\\ 1\le r_{u+1}<\ldots<r_m\le n\\ r_1=p_j\\ \{r_2,\ldots,r_m\}=\{p_1,\ldots,p_m\}\setminus\{p_j\}}} (-1)^{r+\sum\limits_{i=1}^{u} r_i+\sum\limits_{i=u+1}^{m}(r_i+n)} \det[A_{r_1},\ldots,A_{r_m}]$$

$$= \sum_{\substack{1\le r_1<\ldots<r_u\le n\\ 1\le r_{u+1}<\ldots<r_m\le n\\ r_1=p_j\\ \{r_2,\ldots,r_m\}=\{p_1,\ldots,p_m\}\setminus\{p_j\}}} (-1)^{r+\sum\limits_{i=1}^{m} p_i+nv+\ell([p_j,r_2,\ldots,r_m],[p_1,\ldots,p_m])} \det[A_{p_1},\ldots,A_{p_m}],$$

where $\ell([p_j,r_2,\ldots,r_m],[p_1,\ldots,p_m])$ is the number of interchanges of two adjacent columns that should be performed to obtain $[A_{p_1},\ldots,A_{p_m}]$ from $[A_{p_j},A_{r_2},\ldots,A_{r_m}]$. Since

$$\begin{aligned}
&\ell([p_j,r_2,\ldots,r_m],[p_1,\ldots,p_m])\\
&\qquad= \ell([p_j,r_2,\ldots,r_m],[p_1,\ldots,p_j,r_2,\ldots,r_u,r_{u+j},\ldots,r_m])\\
&\qquad\quad+\ell([p_1,\ldots,p_j,r_2,\ldots,r_u,r_{u+j},\ldots,r_m],[p_1,\ldots,p_m])\\
&\qquad= (j-1)u+\ell([r_2,\ldots,r_u,r_{u+j},\ldots,r_m],[p_{j+1},\ldots,p_m]),
\end{aligned}$$

we have

$$\mathcal{S}_j(p_1,\ldots,p_m) = (-1)^{r+\sum\limits_{i=1}^{m} p_i} \det[A_{p_1},\ldots,A_{p_m}](-1)^{nv+(j-1)u} Z_j,$$

where

$$Z_j = \sum_{\substack{1\le r_2<\ldots<r_u\le n\\ 1\le r_{u+j}<\ldots<r_m\le n\\ \{r_2,\ldots,r_u,r_{u+j},\ldots,r_m\}=\{p_{j+1},\ldots,p_m\}}} (-1)^{\ell([r_2,\ldots,r_u,r_{u+j},\ldots,r_m],[p_{j+1},\ldots,p_m])}.$$

The number Z_j depends only on the number of elements in $\{p_{j+1},\ldots,p_m\}$ and does not depend on the numbers $p_1,\ldots,p_m$, therefore

$$\sum_{M\in G_a(A)} (-1)^{r+c(a,M)} \det M = \det A\cdot\sum_{j=1}^{v+1}(-1)^{(n+u)(j-1)}(-1)^{n(v-(j-1))}Z_j.$$

For each $j\in\{1,2,\ldots,v+1\}$ define $\widetilde{r}_j = 1+2+\ldots+(m-j)$ and consider a $(m-j)\times n$ matrix $\widetilde{\mathcal{A}}^{(j)} = [\widetilde{\mathcal{A}}^{(j)}_1,\ldots,\widetilde{\mathcal{A}}^{(j)}_n]$ with columns $\widetilde{\mathcal{A}}^{(j)}_1,\ldots,\widetilde{\mathcal{A}}^{(j)}_n$.

We have

$$
\begin{aligned}
&\det\widetilde{\mathcal{A}}^{(j)} \cdot (-1)^{n(v-(j-1))} Z_j \\
&\quad = \sum_{1\le p_{j+1}<\ldots<p_m\le n} (-1)^{\widetilde{r}_j+\sum\limits_{i=j+1}^{m} p_i} \det[\widetilde{\mathcal{A}}^{(j)}_{p_{j+1}},\ldots,\widetilde{\mathcal{A}}^{(j)}_{p_m}](-1)^{n(v-(j-1))} Z_j \\
&\quad = \sum_{1\le p_{j+1}<\ldots<p_m\le n} \sum_{\substack{1\le r_2<\ldots<r_u\le n\\ 1\le r_{u+j}<\ldots<r_m\le n\\ \{r_2,\ldots,r_u,r_{u+j},\ldots,r_m\}=\{p_{j+1},\ldots,p_m\}}} (-1)^{\widetilde{r}_j+\sum\limits_{i=2}^{u} r_i+\sum\limits_{i=u+j}^{m}(r_i+n)} \\
&\qquad\qquad \times \det[\widetilde{\mathcal{A}}^{(j)}_{r_2},\ldots,\widetilde{\mathcal{A}}^{(j)}_{r_u},\widetilde{\mathcal{A}}^{(j)}_{r_{u+j}},\ldots,\widetilde{\mathcal{A}}^{(j)}_{r_m}] \qquad (5.3)\\
&\quad = \sum_{M\in G_{(u-1,v-j+1)}(\widetilde{\mathcal{A}}^{(j)})} (-1)^{\widetilde{r}_j+c((u-1,v-j+1),M)} \det M \\
&\quad = S^{\widetilde{n}}_{(u-1,v-j+1)} \cdot \det\widetilde{\mathcal{A}}^{(j)}.
\end{aligned}
$$

Finally, applying (5.1f), we obtain

$$
\begin{aligned}
\sum_{M\in G_a(A)} (-1)^{r+c(a,M)} \det M &= \det A \cdot \sum_{j=1}^{v+1} (-1)^{(n+u)(j-1)} S^{\widetilde{n}}_{(u-1,v-j+1)} \\
&= \det A \cdot S^{\widetilde{n}}_a.
\end{aligned}
$$

(g) For an arbitrary integer $k \ge 3$ and $a = (a_1,\ldots,a_p,0,\ldots,0)$, where $p < k$ and $a_1 \ne 0$, $a_p \ne 0$, we have $G_a(A) = G_{(a_1,\ldots,a_p)}(A)$ and $c(a,M) = c((a_1,\ldots,a_p),M)$ for every $M \in G_a(A)$. Therefore, applying (5.1g), we obtain

$$
\begin{aligned}
\sum_{M\in G_a(A)} (-1)^{r+c(a,M)} \det M &= \sum_{M\in G_{(a_1,\ldots,a_p)}(A)} (-1)^{r+c((a_1,\ldots,a_p),M)} \det M \\
&= S^{\widetilde{n}}_{(a_1,\ldots,a_p)} \det A = S^{\widetilde{n}}_a \det A.
\end{aligned}
$$

(h) Let $k \ge 3$ be an integer and $a = (a_1,\ldots,a_k)$, where $p < k$, $a_1 = \ldots = a_p = 0$ and $a_{p+1} \ne 0$. For every matrix $M = [M_1,\ldots,M_m] \in G_a(A)$ such that $M_i = B_{j_i}$ for each $i = 1,2,\ldots,m$, we have $M \in G_{(a_{p+1},\ldots,a_k)}(A)$ and

$$
c(a,M) = \sum_{i=1}^{m} j_i = \sum_{i=1}^{m} (j_i - np) + mnp = c((a_{p+1},\ldots,a_k),M) + mnp.
$$

Since there is a one-to-one correspondence between matrices in $G_a(A)$ and $G_{(a_{p+1},\dots,a_k)}(A)$, we have

$$\begin{aligned}\sum_{M\in G_a(A)} &(-1)^{r+c(a,M)} \det M \\ &= (-1)^{mnp} \sum_{M\in G_{(a_{p+1},\dots,a_k)}(A)} (-1)^{r+c((a_{p+1},\dots,a_k),M)} \det M \\ &= (-1)^{mnp} \cdot S^{\widetilde{n}}_{(a_{p+1},\dots,a_k)} \det A.\end{aligned}$$

Now, applying (5.1h), we get (5.2).

(i) Let $k \geq 3$ be an integer and $a = (a_1, \dots, a_k)$, where $a_1 \neq 0$ and $a_p \neq 0$. Notice that

$$\sum_{M\in G_a(A)} (-1)^{r+c(a,M)} \det M = \sum_{1\leq p_1<\dots<p_m\leq n} \mathcal{S}(p_1, \dots, p_m),$$

where

$$\mathcal{S}(p_1, \dots, p_m) = \sum (-1)^{r+c(a,M)} \det M$$

is the sum over all matrices $M \in G_a(A)$ such that $A_{p_1}, A_{p_2}, \dots, A_{p_m}$ are the columns of M. Let $b = (b_1, \dots, b_{k-1})$, where $b_i = a_i$ for $i = 1, 2, \dots, k-2$ and $b_{k-1} = a_{k-1} + a_k$, For $1 \leq p_1 < \dots < p_m \leq n$, we have

$$\begin{aligned}\mathcal{S}(p_1, \dots, p_m) &= \sum_{\substack{1\leq r_1<\dots<r_{a_1}\leq n \\ 1\leq r_{a_1+1}<\dots<r_{a_1+a_2}\leq n \\ \cdots \\ 1\leq r_{a_1+\dots+a_{k-1}+1}<\dots<r_m\leq n \\ \{r_1,\dots,r_m\}=\{p_1,\dots,p_m\}}} (-1)^{r+\sum\limits_{i=1}^{k} r_i+\sum\limits_{i=2}^{k} a_i(i-1)n} \det[A_{r_1}, \dots, A_{r_m}] \\ &= \sum (-1)^{r+\sum\limits_{i=1}^{m} p_i+\sum\limits_{i=2}^{k} b_i(i-1)n+a_kn} \\ &\qquad \times (-1)^{\ell([r_{b_1+\dots+b_{k-2}+1},\dots,r_m],[\widetilde{p}_{b_1+\dots+b_{k-2}+1},\dots,\widetilde{p}_m])} \\ &\qquad \times \det[A_{r_1}, \dots, A_{r_{b_1+\dots+b_{k-2}}}, A_{\widetilde{p}_{b_1+\dots+b_{k-2}+1}}, \dots, A_{\widetilde{p}_m}],\end{aligned}$$

where the last sum is taken over the indices satisfying the following relations:

$$\begin{aligned}&1 \leq r_1 < \dots < r_{a_1} \leq n, \quad 1 \leq r_{a_1+1} < \dots < r_{a_1+a_2} \leq n, \ \dots\ , \\ &1 \leq r_{a_1+\dots+a_{k-1}+1} < \dots < r_m \leq n, \\ &\{r_1, \dots, r_m\} = \{p_1, \dots, p_m\}, \\ &1 \leq \widetilde{p}_{b_1+\dots+b_{k-2}+1} < \dots < \widetilde{p}_m \leq n, \\ &\{\widetilde{p}_{b_1+\dots+b_{k-2}+1}, \dots, \widetilde{p}_m\} = \{r_{b_1+\dots+b_{k-2}+1}, \dots, r_m\}.\end{aligned}$$

Since the sum

$$Z = \sum_{\substack{1\le \widetilde{p}_{b_1+\ldots+b_{k-2}+1}<\ldots<\widetilde{p}_m\le n \\ \{r_{b_1+\ldots+b_{k-2}+1},\ldots,r_m\}=\{\widetilde{p}_{b_1+\ldots+b_{k-2}+1},\ldots,\widetilde{p}_m\} \\ 1\le r_{b_1+\ldots+b_{k-2}}<\ldots<r_{b_1+\ldots+b_{k-1}}\le n \\ 1\le r_{b_1+\ldots+b_{k-1}+1}<\ldots<r_m\le n}} (-1)^{\ell([r_{b_1+\ldots+b_{k-2}+1},\ldots,r_m],[\widetilde{p}_{b_1+\ldots+b_{k-2}+1},\ldots,\widetilde{p}_m])}$$

does not depend on the numbers $p_1,\ldots,p_m$, $\widetilde{p}_1,\ldots,\widetilde{p}_m$ (it depends only on the number of elements in $\{\widetilde{p}_{b_1+\ldots+b_{k-2}+1},\ldots,\widetilde{p}_m\}$), we have

$$\begin{aligned}\mathcal{S}(p_1,\ldots,p_m) = \sum (-1)^{r+\sum_{i=1}^{k} p_i+\sum_{i=2}^{n-1} b_i(i-1)n} & \\ \times \det[A_{r_1},\ldots,A_{r_{b_1+\ldots+b_{k-2}}},A_{\widetilde{p}_{b_1+\ldots+b_{k-2}+1}},\ldots,A_{\widetilde{p}_m}] & \\ \times (-1)^{a_k n} Z, &\end{aligned}$$

where the sum is taken over the indices satisfying the following relations:

$$\begin{aligned}&1\le r_1<\ldots<r_{b_1}\le n,\quad 1\le r_{b_1+1}<\ldots<r_{b_1+b_2}\le n,\ \ldots\ ,\\ &1\le r_{b_1\ldots+b_{k-3}+1}<\ldots<r_{b_1\ldots+b_{k-2}}\le n,\\ &\{r_1,\ldots,r_m\}=\{p_1,\ldots,p_m\},\\ &1\le \widetilde{p}_{b_1+\ldots+b_{k-2}+1}<\ldots<\widetilde{p}_m\le n,\\ &\{\widetilde{p}_{b_1+\ldots+b_{k-2}+1},\ldots,\widetilde{p}_m\}=\{r_{b_1+\ldots+b_{k-2}+1},\ldots,r_m\}.\end{aligned}$$

Assuming that (5.2) is true for any sequence of $k-1$ nonnegative integers which sum is equal to m, we get

$$\begin{aligned}\sum_{M\in G_a(A)} (-1)^{r+c(a,M)}\det M &= \sum_{M\in G_b(A)} (-1)^{r+c(b,M)}\det M\cdot(-1)^{a_k n} Z\\ &= \det A\cdot S_b^{\widetilde{n}}\cdot(-1)^{a_k n} Z.\end{aligned}$$

For every matrix $M\in G_{(a_k-1,a_k)}$ we have

$$c((a_{k-1},a_k),M) = c((a_{k-1}+a_k),M)+a_k n = c((b_{k-1}),M)+a_k n.$$

Therefore, for every $b_{k-1}\times n$ matrix $\mathcal{A}$, calculations similar to those in (5.3) yield

$$\det\mathcal{A}\cdot(-1)^{a_k n}Z = \det\mathcal{A}\cdot S^{\widetilde{n}}_{(a_{k-1},a_k)}$$

and hence

$$(-1)^{a_k n}Z = S^{\widetilde{n}}_{(a_{k-1},a_k)}.$$

Finally, applying (5.1i), we obtain

$$\begin{aligned}\sum_{M\in G_a(A)} (-1)^{r+c(a,M)}\det M &= \det A\cdot S_b^{\widetilde{n}}\cdot(-1)^{a_k n} Z\\ &= \det A\cdot S_b^{\widetilde{n}}\cdot S^{\widetilde{n}}_{(a_{k-1},a_k)} = \det A\cdot S_a^{\widetilde{n}}.\end{aligned}$$ □

The next two lemmas give explicit formulas for $S^{\widetilde{n}}_{(a_1,\dots,a_k)}$.

Lemma 5.4. *Let n and m be positive integers, $\widetilde{n} = \frac{1-(-1)^n}{2}$ and $m = p+q$, where p, q are nonnegative integers.*

(a) *If $p \not\equiv q \pmod 2$, then*

$$S^{\widetilde{n}}_{(p,q)} = \binom{\frac{p+q-1}{2}}{\lfloor \frac{p}{2} \rfloor}(-1)^{(p+1)n} = \binom{\frac{p+q-1}{2}}{\lfloor \frac{q}{2} \rfloor}(-1)^{qn} = \binom{\frac{p+q-1}{2}}{\lfloor \frac{p}{2} \rfloor, \lfloor \frac{q}{2} \rfloor}(-1)^{qn}. \tag{5.4}$$

(b) *If both p and q are even, then*

$$S^{\widetilde{n}}_{(p,q)} = \binom{\frac{p+q}{2}}{\frac{p}{2}} = \binom{\frac{p+q}{2}}{\frac{q}{2}} = \binom{\frac{p+q}{2}}{\frac{p}{2}, \frac{q}{2}}. \tag{5.5}$$

(c) *If both p and q are odd, then*

$$S^{\widetilde{n}}_{(p,q)} = 0. \tag{5.6}$$

Proof. The proof will be done by induction. If $p < 2$ or $q < 2$, then (5.4), (5.5) and (5.6) follow from (5.1b), (5.1c), (5.1d), (5.1e) and are easy to verify. For $p \geq 2$ and $q \geq 2$, in each of the following cases we apply (5.1f) first.

(a_1) If p is odd and q is even, we have

$$\begin{aligned}
S^{\widetilde{n}}_{(p,q)} &= \binom{\frac{p+q-1}{2}}{\frac{p-1}{2}} + \sum_{j=1}^{q/2}\Big[(-1)^{(2j-1)(p+n)}S^{\widetilde{n}}_{(p-1,q-2j+1)} + (-1)^{2j(p+n)}S^{\widetilde{n}}_{(p-1,q-2j)}\Big] \\
&= \binom{\frac{p+q-1}{2}}{\frac{p-1}{2}} + \sum_{j=1}^{q/2}\left[-\binom{\frac{p+q-1}{2}-j}{\lfloor\frac{p-1}{2}\rfloor} + \binom{\frac{p+q-1}{2}-j}{\frac{p-1}{2}}\right] \\
&= \binom{\frac{p+q-1}{2}}{\lfloor\frac{p-1}{2}\rfloor}(-1)^{(p+1)n}.
\end{aligned}$$

(a_2) If p is even and q is odd, then

$$\begin{aligned}
S^{\widetilde{n}}_{(p,q)} &= \sum_{i=0}^{q}(-1)^{i(p+n)}S^{\widetilde{n}}_{(p-1,q-i)} = \sum_{j=0}^{(q-1)/2}(-1)^{(2j+1)(p+n)}S^{\widetilde{n}}_{(p-1,q-2j-1)} \\
&= \sum_{j=0}^{(q-1)/2}(-1)^n\binom{\frac{p+q-3}{2}-j}{\lfloor\frac{p-1}{2}\rfloor} = \binom{\frac{p+q-3}{2}+1}{\lfloor\frac{p-1}{2}\rfloor+1}(-1)^n \\
&= \binom{\frac{p+q-1}{2}}{\lfloor\frac{p}{2}\rfloor}(-1)^{(p+1)n}.
\end{aligned}$$

(b) If both p and q are even, we obtain

$$S^{\widetilde{n}}_{(p,q)} = \sum_{j=0}^{q/2} (-1)^{2j(p+n)} S^{\widetilde{n}}_{(p-1,q-2j)} = \sum_{j=0}^{q/2} (-1)^{pn} \binom{\frac{p+q-2}{2} - j}{\lfloor \frac{p-1}{2} \rfloor}$$
$$= \binom{\frac{p+q-2}{2} + 1}{\lfloor \frac{p-1}{2} \rfloor + 1} = \binom{\frac{p+q}{2}}{\frac{p}{2}}.$$

(c) If both p and q are odd, we have

$$S^{\widetilde{n}}_{(p,q)} = \sum_{j=0}^{(q-1)/2} \left[(-1)^{2j(p+n)} S^{\widetilde{n}}_{(p-1,q-2j)} + (-1)^{(2j+1)(p+n)} S^{\widetilde{n}}_{(p-1,q-2j-1)} \right]$$
$$= \sum_{j=0}^{(q-1)/2} \left[(-1)^{n} \binom{\frac{p+q-2}{2} - j}{\lfloor \frac{p-1}{2} \rfloor} + (-1)^{n+1} \binom{\frac{p+q-2}{2} - j}{\frac{p-1}{2}} \right] = 0. \qquad \square$$

Remark 5.5. In [4, §33 and §34], Cullis proved the following formula

$$Q^{n-m}_{k-m} \det A = \sum_{1 \le j_1 \le \ldots \le j_k \le n} (-1)^{\sum_{i=1}^{k} (j_i - i)} \det[A_{j_1}, \ldots, A_{j_k}],$$

where $1 \le m \le k \le n$, $A = [A_1, \ldots, A_n]$ is an $m \times n$ matrix and the numbers Q^{n-m}_{k-m} are defined by

$$Q^{2n}_{2m} = Q^{2n+1}_{2m+1} = Q^{2n+1}_{2m} = \binom{n}{m} \quad \text{and} \quad Q^{2n+2}_{2m+1} = 0,$$

where m, n are nonnegative integers and $m \le n$. From Lemma 5.4 it follows that

$$Q^{n-m}_{k-m} = S^{0}_{(k-m,n-k)} = S^{0}_{(n-k,k-m)}.$$

Lemma 5.6. *Let k, m and n be positive integers, $\widetilde{n} = \frac{1-(-1)^n}{2}$ and $m = \sum_{i=1}^{k} a_i$ be a sum of nonnegative integers.*

(A) *If all the numbers $a_1, \ldots, a_k$ are even, then*

$$S^{\widetilde{n}}_{(a_1,\ldots,a_k)} = \binom{\frac{a_1+\ldots+a_k}{2}}{\frac{a_1}{2}, \ldots, \frac{a_k}{2}}. \tag{5.7}$$

(B) *If there is exactly one odd number a_p among $a_1, \ldots, a_k$, then*

$$S^{\widetilde{n}}_{(a_1,\ldots,a_k)} = \binom{\frac{a_1+\ldots+a_k-1}{2}}{\lfloor \frac{a_1}{2} \rfloor, \ldots, \lfloor \frac{a_k}{2} \rfloor} (-1)^{(p+1)n}. \tag{5.8}$$

(C) *If there are at least two odd numbers among $a_1, \ldots, a_k$, then*

$$S^{\widetilde{n}}_{(a_1,\ldots,a_k)} = 0. \tag{5.9}$$

Proof. The proof will be done by induction on k.

For $k=1$, we consider only (5.7) and (5.8). They easily follow from (5.1a). For $k=2$, formulas (5.7), (5.8) and (5.9) follow from Lemma 5.4. (Note that p in (5.4) and p in (5.8) are generally not equal, but they have the same parity if $k=2$.) For $k\geq 3$, consider the following cases.

(A) If all the numbers $a_1,\dots,a_k$ are even, we apply (5.1i) and obtain

$$S^{\widetilde{n}}_{(a_1,\dots,a_k)} = S^{\widetilde{n}}_{(a_1,\dots,a_{k-2},a_{k-1}+a_k)} S^{\widetilde{n}}_{(a_{k-1},a_k)}$$
$$= \binom{\frac{a_1+\dots+a_{k-2}+a_{k-1}+a_k}{2}}{\frac{a_1}{2},\dots,\frac{a_{k-2}}{2},\frac{a_{k-1}+a_k}{2}} \binom{\frac{a_{k-1}+a_k}{2}}{\frac{a_{k-1}}{2},\frac{a_k}{2}} = \binom{\frac{a_1+\dots+a_k}{2}}{\frac{a_1}{2},\dots,\frac{a_k}{2}}.$$

(B) If a_p is odd for some $p\in\{1,2,\dots,k\}$ and a_i is even for each $i\in\{1,2,\dots,k\}\setminus\{p\}$, consider two subcases.

(B_1) If $p<k-1$, we apply (5.1i) and obtain

$$S^{\widetilde{n}}_{(a_1,\dots,a_k)} = \binom{\frac{a_1+\dots+a_k-1}{2}}{\lfloor\frac{a_1}{2}\rfloor,\dots,\lfloor\frac{a_{k-1}+a_k}{2}\rfloor}(-1)^{(p+1)n}\binom{\frac{a_{k-1}+a_k}{2}}{\frac{a_{k-1}}{2},\frac{a_k}{2}}$$
$$= \binom{\frac{a_1+\dots+a_k-1}{2}}{\lfloor\frac{a_1}{2}\rfloor,\dots,\lfloor\frac{a_k}{2}\rfloor}(-1)^{(p+1)n}.$$

(B_2) Similarly, if $p-k=0$ or $p-k=-1$, we have

$$S^{\widetilde{n}}_{(a_1,\dots,a_k)} = \binom{\frac{a_1+\dots+a_k-1}{2}}{\lfloor\frac{a_1}{2}\rfloor,\dots,\lfloor\frac{a_{k-1}+a_k}{2}\rfloor}(-1)^{kn}\binom{\frac{a_{k-1}+a_k-1}{2}}{\lfloor\frac{a_{k-1}}{2}\rfloor,\lfloor\frac{a_k}{2}\rfloor}(-1)^{(p-k+3)n}$$
$$= \binom{\frac{a_1+\dots+a_k-1}{2}}{\lfloor\frac{a_1}{2}\rfloor,\dots,\lfloor\frac{a_k}{2}\rfloor}(-1)^{(p+1)n}.$$

(C) If there are at least two odd numbers among $a_1,\dots,a_k$, let a_p and a_q, where $p<q$, $p,q\in\{1,2,\dots,k\}$, be the two largest odd numbers in $\{a_1,\dots,a_k\}$. Consider two subcases.

(C_1) If $p>1$, we apply (5.1i) $k-p$ times and obtain

$$S^{\widetilde{n}}_{(a_1,\dots,a_k)} = S^{\widetilde{n}}_{(a_1,\dots,a_{p-1},a_p+a_{p+1}+\dots+a_k)} \times S^{\widetilde{n}}_{(a_p,a_{p+1}+\dots+a_k)}\cdots S^{\widetilde{n}}_{(a_{k-2},a_{k-1}+a_k)}S^{\widetilde{n}}_{(a_{k-1},a_k)}.$$

The numbers a_p and $a_{p+1}+\dots+a_n$ are odd, so $S^{\widetilde{n}}_{(a_p,a_{p+1}+\dots+a_n)}=0$ and consequently $S^{\widetilde{n}}_{(a_1,\dots,a_n)}=0$.

(C_2) If $p=1$, then $S^{\widetilde{n}}_{(a_1,a_2+\dots+a_k)}=0$ and after applying (5.1i) $k-2$ times, we have

$$S^{\widetilde{n}}_{(a_1,\dots,a_k)} = S^{\widetilde{n}}_{(a_1,a_2+\dots+a_k)}S^{\widetilde{n}}_{(a_2,a_3+\dots+a_k)}\cdots S^{\widetilde{n}}_{(a_{k-2},a_{k-1}+a_k)}S^{\widetilde{n}}_{(a_{k-1},a_k)}=0. \quad \square$$

Lemma 5.7. *Let k, m, n be positive integers and $\widetilde{n} = \frac{1-(-1)^n}{2}$. Then*
(5.10)
$$\sum_{\substack{(a_1,\ldots,a_k)\\ a_1+\ldots+a_k=m\\ a_i\in\mathbb{Z},\ a_i\geq 0,\ i=1,2,\ldots,k}} S^{\widetilde{n}}_{(a_1,\ldots,a_k)} = \begin{cases} k^{m/2}, & \text{if } m \text{ is even},\\ k^{(m+1)/2}, & \text{if } m \text{ is odd and } n \text{ is even},\\ 0, & \text{if } m, n \text{ are odd and } k \text{ is even},\\ k^{(m-1)/2}, & \text{if } m, n, k \text{ are odd}.\end{cases}$$

Proof. Denote the left-hand side of (5.10) by L and consider two cases.

(i) If m is even, we apply Lemma 5.6 (C) and Lemma 5.6 (A), and obtain

$$L = \sum_{\substack{(a_1,\ldots,a_k)\\ a_1+\ldots+a_k=m\\ a_i\geq 0,\ a_i\equiv 0 \pmod 2,\ i=1,2,\ldots,k}} S^{\widetilde{n}}_{(a_1,\ldots,a_k)} = \sum_{\substack{(a_1,\ldots,a_k)\\ a_1+\ldots+a_k=m\\ a_i\geq 0,\ a_i\equiv 0 \pmod 2,\ i=1,2,\ldots,k}} \binom{\frac{a_1+\ldots+a_k}{2}}{\frac{a_1}{2},\ldots,\frac{a_k}{2}} = k^{\frac{m}{2}},$$

(ii) If m is odd, by $\mathcal{A}$ we denote the set of all sequences of nonnegative integers $(a_1,\ldots,a_k)$ containing exactly one odd number and satisfying $\sum_{i=1}^k a_i = m$. By Lemma 5.6 (B), we have

$$\begin{aligned} L &= \sum_{p=1}^{k} \sum_{\substack{(a_1,\ldots,a_k)\in\mathcal{A}\\ a_p\equiv 1 \pmod 2}} \binom{\frac{a_1+\ldots+a_k-1}{2}}{\lfloor\frac{a_1}{2}\rfloor,\ldots,\lfloor\frac{a_k}{2}\rfloor}(-1)^{(p+1)n}\\ &= \sum_{p=1}^{k} \sum_{\substack{(a_1,\ldots,a_k)\in\mathcal{A}\\ a_p\equiv 1 \pmod 2}} \binom{\frac{a_1+\ldots+a_{p-1}+(a_p-1)+a_{p+1}+\ldots+a_k}{2}}{\frac{a_1}{2},\ldots,\frac{a_{p-1}}{2},\frac{(a_p-1)}{2},\frac{a_{p+1}}{2},\ldots,\frac{a_k}{2}}(-1)^{(p+1)n}\\ &= \begin{cases} k\cdot k^{(m-1)/2}, & \text{if } n \text{ is even},\\ \sum\limits_{p=1}^{k}(-1)^{p+1}\cdot k^{(m-1)/2}, & \text{if } n \text{ is odd},\end{cases}\\ &= \begin{cases} k^{(m+1)/2}, & \text{if } n \text{ is even},\\ 0, & \text{if } n \text{ is odd and } k \text{ is even},\\ k^{(m-1)/2}, & \text{if } n \text{ is odd and } k \text{ is odd}.\end{cases} \end{aligned}$$

□

Example 5.8. If $A = [A_1, A_2, A_3]$ is a 3×3 matrix and

$$B = [A, A, A] = [A_1, A_2, A_3, A_1, A_2, A_3, A_1, A_2, A_3]$$

is a 3×9 matrix, then

$$\det B = 3\det A.$$

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