doi: 10.17951/a.2021.75.1.15-36

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VOL. LXXV, NO. 1, 2021 SECTIO A 15–36

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Weighted integral inequalities related to Wirtinger's result for p-norms with applications

Abstract. In this paper we establish several natural consequences of some Wirtinger type integral inequalities for p-norms. The corresponding weighted versions and applications related to the weighted trapezoid inequalities, to weighted Grüss' type inequalities and reverses of Jensen's inequality are also provided.

1. Introduction. The following Wirtinger type inequalities are well known

$$\int_a^b u^2(t)dt \leq \frac{(b-a)^2}{\pi^2} \int_a^b \left[u'(t)\right]^2 dt, \tag{1.1}$$

provided $u \in C^1([a,b],\mathbb{R})$ and $u(a) = u(b) = 0$ with the equality holding if and only if $u(t) = K \sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant K, and, similarly, if $u \in C^1([a,b],\mathbb{R})$ satisfies $u(a) = 0$, then

$$\int_a^b u^2(t)dt \leq \frac{4(b-a)^2}{\pi^2} \int_a^b \left[u'(t)\right]^2 dt. \tag{1.2}$$

The equality holds in (1.2) if and only if $u(t) = K \sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant K.

2010 *Mathematics Subject Classification.* 26D15, 26D10.

Key words and phrases. Wirtinger's inequality, trapezoid inequality, Grüss' inequality, Jensen's inequality.

For $p > 1$ put $\pi_{p-1} := \frac{2\pi}{p}\sin\left(\frac{\pi}{p}\right)$. In [12], J. Jaroš obtained, as a particular case of a more general inequality, the following generalization of (1.1):

$$\int_a^b |u(t)|^p\,dt \le \frac{(b-a)^p}{(p-1)\,\pi_{p-1}^p}\int_a^b |u'(t)|^p\,dt, \tag{1.3}$$

provided $u \in C^1([a,b],\mathbb{R})$ and $u(a) = u(b) = 0$, with the equality if and only if $u(t) = K\sin_{p-1}\left[\frac{\pi_{p-1}(t-a)}{b-a}\right]$ for some $K \in \mathbb{R}$, where $\sin_{p-1}$ is the $2\pi_{p-1}$-periodic generalized sine function, see [19] or [6].

If $u(a) = 0$ and $u \in C^1([a,b],\mathbb{R})$, then

$$\int_a^b |u(t)|^p\,dt \le \frac{[2\,(b-a)]^p}{(p-1)\,\pi_{p-1}^p}\int_a^b |u'(t)|^p\,dt \tag{1.4}$$

with the equality if and only if $u(t) = K\sin_{p-1}\left[\frac{\pi_{p-1}(t-a)}{2(b-a)}\right]$ for some $K \in \mathbb{R}$.

The inequalities (1.3) and (1.4) were obtained for $a = 0$, $b = 1$ and $q = p > 1$ in [18] by the use of an elementary proof.

For some related Wirtinger type integral inequalities see [1,2,5,10,12,13] and [16–18].

In this paper we establish several natural consequences of the Wirtinger type integral inequalities for p-norms from (1.3) and in (1.4). The corresponding weighted versions and applications related to the weighted trapezoid inequalities, to weighted Grüss' type inequalities and reverses of Jensen's inequality are also provided.

2. Some weighted inequalities.

We have:

Theorem 1. *Let $g : [a,b] \to [g(a), g(b)]$ be a continuous strictly increasing function of class C^1 on (a,b) and $p > 1$.*

(i) *If $f \in C^1([a,b],\mathbb{R})$ is such $f(a) = f(b) = 0$, then*

$$\int_a^b |f(t)|^p\,g'(t)dt \le \frac{[g(b)-g(a)]^p}{(p-1)\,\pi_{p-1}^p}\int_a^b \frac{|f'(t)|^p}{|g'(t)|^{p-1}}dt. \tag{2.1}$$

The equality holds in (2.1) if and only if

$$f(t) = K\sin_{p-1}\left[\frac{\pi_{p-1}\,(g(t)-g(a))}{g(b)-g(a)}\right],\ K \in \mathbb{R}.$$

(ii) *If $f \in C^1([a,b],\mathbb{R})$ is such that $f(a) = 0$, then*

$$\int_a^b |f(t)|^p\,g'(t)dt \le \frac{2^p\,[g(b)-g(a)]^p}{(p-1)\,\pi_{p-1}^p}\int_a^b \frac{|f'(t)|^p}{|g'(t)|^{p-1}}dt. \tag{2.2}$$

The equality holds in (2.2) if and only if

$$f(t) = K\sin_{p-1}\left[\frac{\pi_{p-1}\,(g(t)-g(a))}{2\,(g(b)-g(a))}\right],\ K \in \mathbb{R}.$$

Proof. (i) We write the inequality (1.3) for the function $u = f \circ g^{-1}$ on the interval $[g(a), g(b)]$ to get

$$\int_{g(a)}^{g(b)} \left| f \circ g^{-1}(z) \right|^p dz \leq \frac{(g(b) - g(a))^p}{(p-1)\,\pi_{p-1}^p} \int_{g(a)}^{g(b)} \left| \left(f \circ g^{-1} \right)'(z) \right|^p dz. \tag{2.3}$$

If $f\colon [c, d] \to \mathbb{R}$ is absolutely continuous on $[c, d]$, then $f \circ g^{-1}\colon [g(c), g(d)] \to \mathbb{R}$ is absolutely continuous on $[g(c), g(d)]$ and using the chain rule and the derivative of inverse functions, we have

$$\left(f \circ g^{-1} \right)'(z) = \left(f' \circ g^{-1} \right)(z) \left(g^{-1} \right)'(z) = \frac{\left(f' \circ g^{-1} \right)(z)}{\left(g' \circ g^{-1} \right)(z)} \tag{2.4}$$

for almost every (a.e.) $z \in [g(c), g(d)]$.

Using the inequality (2.3), we then get

$$\int_{g(a)}^{g(b)} \left| f \circ g^{-1}(z) \right|^p dz \leq \frac{(g(b) - g(a))^p}{(p-1)\,\pi_{p-1}^p} \int_{g(a)}^{g(b)} \left| \frac{\left(f' \circ g^{-1} \right)(z)}{\left(g' \circ g^{-1} \right)(z)} \right|^p dz$$

provided $\left(f \circ g^{-1} \right)(g(a)) = f(a) = 0$ and $\left(f \circ g^{-1} \right)(g(b)) = f(b) = 0$.

Observe also that, by the change of variable $t = g^{-1}(z)$, $z \in [g(a), g(b)]$, we have $z = g(t)$ which gives $dz = g'(t)dt$ and

$$\int_{g(a)}^{g(b)} \left| f \circ g^{-1}(z) \right|^p dz = \int_a^b |f(t)|^p \, g'(t) dt.$$

We also have

$$\int_{g(a)}^{g(b)} \left| \frac{\left(f' \circ g^{-1} \right)(z)}{\left(g' \circ g^{-1} \right)(z)} \right|^p dz = \int_a^b \left| \frac{f'(t)}{g'(t)} \right|^p g'(t) dt = \int_a^b \frac{|f'(t)|^p}{|g'(t)|^{p-1}} dt. \tag{2.5}$$

By making use of (2.5), we get (2.1).

The case of equality follows from the case of equality in (1.3).

(ii) Follows in a similar way from (1.4). □

Remark 1. For $p = 2$ we get from Theorem 1 the weighted integral inequalities from [8].

Assume that $p > 1$. Some examples are as follows:

a) If we take $g : [a, b] \subset (0, \infty) \to \mathbb{R}$, $g(t) = \ln t$ and assume that $f \in C^1([a, b], \mathbb{R})$ is such that $f(a) = f(b) = 0$, then by (2.1) we get

$$\int_a^b \frac{|f(t)|^p}{t} dt \leq \frac{\left[\ln\left(\frac{b}{a} \right) \right]^p}{(p-1)\,\pi_{p-1}^p} \int_a^b \left| f'(t) \right|^p t^{p-1} dt. \tag{2.6}$$

The equality holds in (2.6) if and only if

$$f(t) = K \sin_{p-1} \left[\frac{\pi_{p-1} \ln\left(\frac{t}{a} \right)}{\ln\left(\frac{b}{a} \right)} \right], \; K \in \mathbb{R}.$$

b) If we take $g : [a,b] \subset \mathbb{R} \to (0,\infty)$, $g(t) = \exp t$ and assume that $f \in C^1([a,b],\mathbb{R})$ is a function such that $f(a) = f(b) = 0$, then by (2.1) we get

$$\int_a^b |f(t)|^p \exp(t)dt \le \frac{(\exp b - \exp a)^p}{(p-1)\pi_{p-1}^p} \int_a^b \left|f'(t)\right|^p \exp((1-p)t)dt. \tag{2.7}$$

The equality holds in (2.6) if and only if

$$f(t) = K \sin_{p-1}\left[\frac{\pi_{p-1}(\exp t - \exp a)}{\exp b - \exp a}\right],\ K \in \mathbb{R}.$$

c) If we take $g : [a,b] \subset (0,\infty) \to \mathbb{R}$, $g(t) = t^r$, $r > 0$ and assume that $f \in C^1([a,b],\mathbb{R})$ is such that $f(a) = f(b) = 0$, then by (2.1) we get

$$\int_a^b |f(t)|^p t^{r-1}dt \le \frac{(b^r - a^r)^p}{r^p (p-1) \pi_{p-1}^p} \int_a^b \frac{|f'(t)|^p}{t^{(r-1)(p-1)}}dt. \tag{2.8}$$

The equality holds in (2.8) if and only if

$$f(t) = K \sin_{p-1}\left[\frac{\pi_{p-1}(t^r - a^r)}{b^r - a^r}\right],\ K \in \mathbb{R}.$$

If $w : [a,b] \to \mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W : [a,b] \to [0,\infty)$, $W(x) := \int_a^x w(s)ds$ is strictly increasing and differentiable on (a,b). We have $W'(x) = w(x)$ for any $x \in (a,b)$.

Corollary 1. *Assume that* $w : [a,b] \to (0,\infty)$ *is continuous on* $[a,b]$ *and* $f \in C^1([a,b],\mathbb{R})$ *is a function such that* $f(a) = f(b) = 0$, *then*

$$\int_a^b |f(t)|^p w(t)dt \le \frac{1}{(p-1)\pi_{p-1}^p}\left(\int_a^b w(s)ds\right)^p \int_a^b \frac{|f'(t)|^p}{w^{p-1}(t)}dt. \tag{2.9}$$

The equality holds in (2.9) *if and only if*

$$f(t) = K \sin_{p-1}\left[\frac{\pi \int_a^t w(s)ds}{\int_a^b w(s)ds}\right],\ K \in \mathbb{R}.$$

If $f(a) = 0$, *then*

$$\int_a^b |f(t)|^p w(t)dt \le \frac{2^p}{(p-1)\pi_{p-1}^p}\left(\int_a^b w(s)ds\right)^p \int_a^b \frac{|f'(t)|^p}{w^{p-1}(t)}dt \tag{2.10}$$

with the equality if and only if

$$f(t) = K \sin_{p-1}\left[\frac{\pi \int_a^t w(s)ds}{2\int_a^b w(s)ds}\right],\ K \in \mathbb{R}.$$

Remark 2. If f is a function such that $f(a)=0$, then the inequality (2.10) can be stated on the infinite interval $[a,\infty)$ as follows

$$\int_a^\infty |f(t)|^p\, w(t)dt \le \frac{2^p}{(p-1)\,\pi_{p-1}^p}\left(\int_a^\infty w(s)ds\right)^p \int_a^\infty \frac{|f'(t)|^p}{w^{p-1}(t)}dt, \tag{2.11}$$

provided $f\in C^1([a,\infty),\mathbb{R})$, $w:[a,\infty)\to(0,\infty)$ is continuous on $[a,\infty)$ and the integrals above exist. The equality holds if and only if

$$f(t)=K\sin_{p-1}\left[\frac{\pi\int_a^t w(s)ds}{2\int_a^\infty w(s)ds}\right],\ K\in\mathbb{R}.$$

3. Some weighted inequalities of trapezoid type. We have:

Theorem 2. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *and* $g\in C^1([a,b],\mathbb{R})$, *then for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}&\left|\frac{1}{\int_a^b w(t)dt}\int_a^b \frac{w(t)+w(a+b-t)}{2}g(t)dt-\frac{g(a)+g(b)}{2}\right|\\ &\le \frac{1}{2\,(p-1)^{1/p}\,\pi_{p-1}}\left(\int_a^b w(s)ds\right)^{1/q}\\ &\quad\times\left(\int_a^b \frac{|g'(t)-g'(a+b-t)|^p}{w^{p-1}(t)}dt\right)^{1/p}\\ &\le \frac{1}{2\,(p-1)^{1/p}\,\pi_{p-1}}\max_{t\in[a,b]}\left|g'(t)-g'(a+b-t)\right|\\ &\quad\times\left(\int_a^b w(s)ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p}.\end{aligned}\tag{3.1}$$

In particular, if w *is symmetrical, i.e.,* $w(a+b-t)=w(t)$ *for any* $t\in[a,b]$, *then we have*

$$\begin{aligned}&\left|\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)g(t)dt-\frac{g(a)+g(b)}{2}\right|\\ &\le \frac{1}{2\,(p-1)^{1/p}\,\pi_{p-1}}\left(\int_a^b w(s)ds\right)^{1/q}\\ &\quad\times\left(\int_a^b \frac{|g'(t)-g'(a+b-t)|^p}{w^{p-1}(t)}dt\right)^{1/p}\\ &\le \frac{1}{2\,(p-1)^{1/p}\,\pi_{p-1}}\max_{t\in[a,b]}\left|g'(t)-g'(a+b-t)\right|\\ &\quad\times\left(\int_a^b w(s)ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p}.\end{aligned}\tag{3.2}$$

Proof. Considering the function

$$f(t) := \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2},\ t \in [a,b],$$

we have $f(a) = f(b) = 0$ and by (2.9) we get

$$\int_a^b \left| \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right|^p w(t)dt \leq \frac{1}{2^p (p-1) \pi_{p-1}^p} \left(\int_a^b w(s)ds \right)^p \int_a^b \frac{|g'(t) - g'(a+b-t)|^p}{w^{p-1}(t)} dt,$$

namely

$$\left(\int_a^b \left| \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right|^p w(t)dt \right)^{1/p} \leq \frac{1}{2(p-1)^{1/p} \pi_{p-1}} \left(\int_a^b w(s)ds \right) \times \left(\int_a^b \frac{|g'(t) - g'(a+b-t)|^p}{w^{p-1}(t)} dt \right)^{1/p}. \tag{3.3}$$

By the weighted Hölder's integral inequality for $p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$ we have

$$\begin{aligned} &\left(\int_a^b \left| \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right|^p w(t)dt \right)^{1/p} \left(\int_a^b w(s)ds \right)^{1/q} \\ &\geq \left| \int_a^b \left[\frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right] w(t)dt \right| \\ &= \left| \int_a^b \frac{g(t)+g(a+b-t)}{2} w(t)dt - \frac{g(a)+g(b)}{2} \int_a^b w(t)dt \right|. \end{aligned} \tag{3.4}$$

Making use of (3.3) and (3.4), we get

$$\begin{aligned} &\left| \int_a^b \frac{g(t)+g(a+b-t)}{2} w(t)dt - \frac{g(a)+g(b)}{2} \int_a^b w(t)dt \right| \\ &\leq \frac{1}{2(p-1)^{1/p} \pi_{p-1}} \left(\int_a^b w(s)ds \right)^{1+1/q} \\ &\quad \times \left(\int_a^b \frac{|g'(t) - g'(a+b-t)|^p}{w^{p-1}(t)} dt \right)^{1/p}. \end{aligned} \tag{3.5}$$

Observe that, by the change of variable $s = a+b-t$, $t \in [a,b]$ we have

$$\int_a^b g(a+b-t)w(t)dt = \int_a^b g(s)w(a+b-s)ds$$

and then

$$\int_a^b \frac{g(t)+g(a+b-t)}{2} w(t)dt = \int_a^b \frac{w(t)+w(a+b-t)}{2} g(t)dt.$$

Utilising (3.5), we get the first part of (3.1). The second inequality in (3.1) is obvious. □

In 1906, Fejér [9], while studying trigonometric polynomials, obtained the following inequalities which generalize those of Hermite & Hadamard:

Theorem 3 (Fejér's inequality). *Consider the integral* $\int_a^b h(x)w(x)dx$, *where* h *is a convex function in the interval* (a,b) *and* w *is a positive function in the same interval such that*

$$w(x) = w(a+b-x)$$

for any $x \in [a,b]$, *i.e.,* $y = w(x)$ *is a symmetric curve with respect to the straight line which contains the point* $\left(\frac{1}{2}(a+b), 0\right)$ *and is normal to the* x*-axis. Under these conditions the following inequalities are valid:*

$$h\left(\frac{a+b}{2}\right) \le \frac{1}{\int_a^b w(x)dx} \int_a^b h(x)w(x)dx \le \frac{h(a)+h(b)}{2}. \tag{3.6}$$

If h *is concave on* (a,b)*, then the inequalities reverse in* (3.6).

Remark 3. If $g : [a,b] \to \mathbb{R}$ is differentiable convex, $g'_-(b)$ and $g'_+(a)$ are finite and $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and symmetrical, then by (3.2) we get the following reverse of the second inequality in (3.6):

$$\begin{aligned} 0 &\le \frac{g(a)+g(b)}{2} - \frac{1}{\int_a^b w(t)dt} \int_a^b w(t)g(t)dt \\ &\le \frac{1}{2(p-1)^{1/p}\pi_{p-1}} \left[g'_-(b) - g'_+(a)\right] \\ &\quad \times \left(\int_a^b w(s)ds\right)^{1/q} \left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p}, \end{aligned} \tag{3.7}$$

provided p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\int_a^b \frac{1}{w^{p-1}(t)}dt < \infty$.

Remark 4. Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and $g \in C^1([a,b],\mathbb{R})$ is a function such that g' is K-Lipschitzian on $[a,b]$, i.e.,

$|g'(t) - g'(s)| \le K|t-s|$ for any $[a,b]$, then by (3.1) we get

$$\left|\frac{1}{\int_a^b w(t)dt}\int_a^b \frac{w(t)+w(a+b-t)}{2}g(t)dt - \frac{g(a)+g(b)}{2}\right|$$

$$\le \frac{1}{(p-1)^{1/p}\pi_{p-1}}K\left(\int_a^b w(s)ds\right)^{1/q}\left(\int_a^b \frac{\left|t-\frac{a+b}{2}\right|^p}{w^{p-1}(t)}dt\right)^{1/p} \tag{3.8}$$

$$\le \frac{b-a}{2(p-1)^{1/p}\pi_{p-1}}K\left(\int_a^b w(s)ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p},$$

provided $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\int_a^b \frac{1}{w^{p-1}(t)}dt<\infty$.

If $g:[a,b]\to\mathbb{R}$ is twice differentiable and convex with $\|g''\|_{[a,b],\infty}<\infty$ and $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and symmetrical, then by (3.8) we get the following reverse of the second inequality in (3.6):

$$0\le \frac{g(a)+g(b)}{2} - \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)g(t)dt$$

$$\le \frac{1}{(p-1)^{1/p}\pi_{p-1}}\left\|g''\right\|_{[a,b],\infty}\left(\int_a^b w(s)ds\right)^{1/q}\left(\int_a^b \frac{\left|t-\frac{a+b}{2}\right|^p}{w^{p-1}(t)}dt\right)^{1/p} \tag{3.9}$$

$$\le \frac{b-a}{2(p-1)^{1/p}\pi_{p-1}}\left\|g''\right\|_{[a,b],\infty}\left(\int_a^b w(s)ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p},$$

provided p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\int_a^b \frac{1}{w^{p-1}(t)}dt<\infty$.

Another trapezoid type weighted inequality is as follows:

Theorem 4. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and $g\in C^1([a,b],\mathbb{R})$, then*

$$\left|\frac{g(a)\left[b-E(w;[a,b])\right]+g(b)\left[E(w;[a,b])-a\right]}{b-a}\right.$$

$$\left.-\frac{1}{\int_a^b w(s)ds}\int_a^b g(t)w(t)dt\right|$$

$$\le \frac{1}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b w(s)ds\right)^{1/q}\left(\int_a^b \frac{\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^p}{w^{p-1}(t)}dt\right)^{1/p} \tag{3.10}$$

$$\le \frac{1}{\pi_{p-1}}\max_{t\in[a,b]}\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|$$

$$\times\left(\int_a^b w(s)ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p},$$

provided p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\int_a^b \frac{1}{w^{p-1}(t)} dt < \infty$, *where*

$$E(w;[a,b]) := \frac{1}{\int_a^b w(s)ds} \int_a^b tw(t)dt.$$

Proof. If $g \in C^1([a,b],\mathbb{R})$, then by taking

$$f(t) := g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a},\ t \in [a,b],$$

we have $f(a) = f(b) = 0$ and by (2.9) we have

$$\int_a^b \left| g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right|^p w(t)dt \le \frac{1}{(p-1)\pi_{p-1}^p} \left(\int_a^b w(s)ds \right)^p \int_a^b \frac{\left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^p}{w^{p-1}(t)} dt,$$

namely

$$\left(\int_a^b \left| g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right|^p w(t)dt \right)^{1/p} \le \frac{1}{(p-1)^{1/p} \pi_{p-1}} \left(\int_a^b w(s)ds \right) \left(\int_a^b \frac{\left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^p}{w^{p-1}(t)} dt \right)^{1/p}. \tag{3.11}$$

By the weighted Hölder's integral inequality for p, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\begin{aligned}
&\left(\int_a^b \left| g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right|^p w(t)dt \right)^{1/p} \left(\int_a^b w(s)ds \right)^{1/q} \\
&\ge \left| \int_a^b g(t)w(t)dt - \int_a^b \frac{g(a)(b-t) + g(b)(t-a)}{b-a} w(t)dt \right| \\
&= \left| \int_a^b g(t)w(t)dt - \frac{1}{b-a} g(a) \int_a^b (b-t)w(t)dt - g(b) \int_a^b (t-a)w(t)dt \right| \\
&= \left| \int_a^b g(t)w(t)dt - \frac{1}{b-a} g(a) \int_a^b (b-t)w(t)dt - g(b) \int_a^b (t-a)w(t)dt \right| \\
&= \left(\int_a^b w(s)ds \right) \left| \frac{1}{\int_a^b w(s)ds} \int_a^b g(t)w(t)dt \right. \\
&\qquad \left. - \frac{g(a)\left[b - E(w;[a,b])\right] + g(b)\left[E(w;[a,b]) - a\right]}{b-a} \right|.
\end{aligned} \tag{3.12}$$

By making use of (3.11) and (3.12), we have

$$\left(\int_a^b w(s)ds\right)\left|\frac{1}{\int_a^b w(s)ds}\int_a^b g(t)w(t)dt - \frac{g(a)\left[b-E\left(w;[a,b]\right)\right]+g(b)\left[E\left(w;[a,b]\right)-a\right]}{b-a}\right| \le \frac{1}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b w(s)ds\right)^{1+1/q}\left(\int_a^b \frac{\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^p}{w^{p-1}(t)}dt\right)^{1/p},$$

which is equivalent to the first inequality in (3.10).

The second inequality in (3.10) is obvious. □

The case of a convex function is as follows:

Corollary 2. *If $g:[a,b]\to\mathbb{R}$ is continuously differentiable convex and $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, then*

$$\begin{aligned} 0 &\le \frac{g(a)\left[b-E\left(w;[a,b]\right)\right]+g(b)\left[E\left(w;[a,b]\right)-a\right]}{b-a} - \frac{1}{\int_a^b w(s)ds}\int_a^b g(t)w(t)dt \\ &\le \frac{1}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b w(s)ds\right)^{1/q} \times\left(\int_a^b \frac{\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^p}{w^{p-1}(t)}dt\right)^{1/p} \\ &\le \frac{1}{\pi_{p-1}}\max_{t\in[a,b]}\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right| \times\left(\int_a^b w(s)ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p}, \end{aligned} \tag{3.13}$$

provided $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\int_a^b \frac{1}{w^{p-1}(t)}dt<\infty$.

The positivity follows by the fact that for a convex function g on $[a,b]$ we have

$$\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\ge g(t)$$

for any $t\in[a,b]$. The rest is obvious by Theorem 4.

4. Some inequalities for the weighted Čebyšev functional. Consider now the *weighted Čebyšev functional*

$$
\begin{aligned}
C_w(f,g) :=& \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)g(t)dt \\
&- \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)dt \cdot \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)g(t)dt,
\end{aligned} \tag{4.1}
$$

where $f,\ g,\ w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)dt > 0$.

In [3], Cerone and Dragomir obtained, among others, the following inequalities:

$$
\begin{aligned}
&|C_w(f,g)| \\
&\leq \frac{M-m}{2}\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)\left|g(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b w(s)g(s)ds\right|dt \\
&\leq \frac{M-m}{2}\left[\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)\left|g(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b w(s)g(s)ds\right|^p dt\right]^{\frac{1}{p}} \\
&\leq \frac{1}{2}(M-m)\operatorname*{essup}_{t\in[a,b]}\left|g(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b w(s)g(s)ds\right|
\end{aligned} \tag{4.2}
$$

for $p > 1$, provided $-\infty < m \leq f(t) \leq M < \infty$ for a.e. $t \in [a,b]$ and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (4.2) in the sense that it cannot be replaced by a smaller constant.

In addition, if $-\infty < n \leq g(t) \leq N < \infty$ for a.e. $t \in [a,b]$, then the following refinement of the celebrated Grüss inequality is obtained:

$$
\begin{aligned}
&|C_w(f,g)| \\
&\leq \frac{M-m}{2}\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)\left|g(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b w(s)g(s)ds\right|dt \\
&\leq \frac{M-m}{2}\left[\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)\left|g(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b w(s)g(s)ds\right|^2 dt\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}(M-m)(N-n).
\end{aligned} \tag{4.3}
$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

We have:

Theorem 5. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$, $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, $f\in L_p([a,b],\mathbb{R})$ *and* $g\in C^1([a,b],\mathbb{R})$, *then*

$$\begin{aligned}|C_w(f,g)| &\le \frac{b-a}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b |g'(t)|^q dt\right)^{1/q} \\ &\times \frac{1}{\int_a^b w(s)ds}\left(\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds\right|^p w^p(t)dt\right)^{1/p}.\end{aligned}\tag{4.4}$$

In particular, if $p=q=2$, *then we have* [8]*:*

$$\begin{aligned}|C_w(f,g)| &\le \frac{b-a}{\pi}\left(\int_a^b |g'(t)|^2 dt\right)^{1/2} \\ &\times \frac{1}{\int_a^b w(s)ds}\left(\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds\right|^2 w^2(t)dt\right)^{1/2}.\end{aligned}\tag{4.5}$$

Proof. Integrating by parts, we have

$$\begin{aligned}&\frac{1}{\int_a^b w(s)ds}\int_a^b\left(\int_a^x f(t)w(t)dt-\frac{\int_a^x w(s)ds}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds\right)g\prime(x)dx \\ &=\frac{1}{\int_a^b w(s)ds}\left[\left(\int_a^x f(t)w(t)dt-\frac{\int_a^x w(s)ds}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds\right)g(x)\Big|_a^b\right. \\ &\quad\left.-\int_a^b g(x)\left(f(x)w(x)-\frac{w(x)}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds\right)dx\right] \\ &=-\frac{1}{\int_a^b w(s)ds}\int_a^b f(x)g(x)w(x)dx \\ &\quad+\frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds\cdot\frac{1}{\int_a^b w(s)ds}\int_a^b g(x)w(x)dx,\end{aligned}$$

which gives

$$\begin{aligned}C_w(f,g) &= \frac{1}{\int_a^b w(s)ds} \\ &\times\int_a^b\left(\frac{\int_a^x w(s)ds}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds-\int_a^x f(t)w(t)dt\right)g'(x)dx.\end{aligned}\tag{4.6}$$

Using the Hölder integral inequality for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we get

$$
\begin{aligned}
|C_w(f,g)| &\leq \frac{1}{\int_a^b w(s)ds} \\
&\times \int_a^b \left| \left(\frac{\int_a^x w(s)ds}{\int_a^b w(s)ds} \int_a^b f(s)w(s)ds - \int_a^x f(t)w(t)dt \right) g'(x) \right| dx \\
&\leq \frac{1}{\int_a^b w(s)ds} \left(\int_a^b \left| \frac{\int_a^x w(s)ds}{\int_a^b w(s)ds} \int_a^b f(s)w(s)ds - \int_a^x f(t)w(t)dt \right|^p dx \right)^{1/p} \\
&\times \left(\int_a^b |g'(x)|^q dx \right)^{1/q}.
\end{aligned} \tag{4.7}
$$

If we take

$$
u(x) := \frac{\int_a^x w(s)ds}{\int_a^b w(s)ds} \int_a^b f(s)w(s)ds - \int_a^x f(t)w(t)dt, \ x \in [a,b],
$$

we observe that $u(a)=u(b)=0$ and $u \in C^1([a,b],\mathbb{R})$.

Using the inequality (1.3), we then have

$$
\begin{aligned}
&\int_a^b \left| \frac{\int_a^x w(s)ds}{\int_a^b w(s)ds} \int_a^b f(s)w(s)ds - \int_a^x f(t)w(t)dt \right|^p dx \\
&\leq \frac{(b-a)^p}{(p-1)\pi_{p-1}^p} \int_a^b \left| \frac{w(x)}{\int_a^b w(s)ds} \int_a^b f(s)w(s)ds - f(x)w(x) \right|^p dx \\
&= \frac{(b-a)^p}{(p-1)\pi_{p-1}^p} \int_a^b \left| \frac{1}{\int_a^b w(s)ds} \int_a^b f(s)w(s)ds - f(x) \right|^p w^p(x)dx,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\left(\int_a^b \left| \frac{\int_a^x w(s)ds}{\int_a^b w(s)ds} \int_a^b f(s)w(s)ds - \int_a^x f(t)w(t)dt \right|^p dx \right)^{1/p} \\
&\leq \frac{b-a}{(p-1)^{1/p}\pi_{p-1}} \\
&\times \left(\int_a^b \left| \frac{1}{\int_a^b w(s)ds} \int_a^b f(s)w(s)ds - f(x) \right|^p w^p(x)dx \right)^{1/p}.
\end{aligned} \tag{4.8}
$$

By using (4.7) and (4.8), we get the desired result (4.4). □

By taking $w \equiv 1$, we get the following unweighted inequality:

Corollary 3. *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $f\in L_p([a,b],\mathbb{R})$ and $g\in C^1([a,b],\mathbb{R})$, then*

$$\begin{aligned}|C(f,g)| &\leq \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b |g'(t)|^q\,dt\right)^{1/q} \\ &\times\left(\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)ds\right|^p dt\right)^{1/p}.\end{aligned} \tag{4.9}$$

In particular, if $p=q=2$, then we have [8]*:*

$$\begin{aligned}|C(f,g)| &\leq \frac{\sqrt{b-a}}{\pi}\left(\int_a^b |g'(t)|^2\,dt\right)^{1/2} \\ &\times\left(\frac{1}{b-a}\int_a^b |f(t)|^2\,dt-\left|\frac{1}{b-a}\int_a^b f(s)ds\right|^2\right)^{1/2}.\end{aligned} \tag{4.10}$$

The following result also holds:

Theorem 6. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $f\in L_p([a,b],\mathbb{R})$ and $g\in C^1([a,b],\mathbb{R})$, then*

$$\begin{aligned}|C_w(f,g)| &\leq \frac{2}{(q-1)^{1/q}\,\pi_{q-1}}\left(\int_a^b w(s)ds\right)^{1/p}\left(\int_a^b \frac{|g'(t)|^q}{w^{q-1}(t)}dt\right)^{1/q} \\ &\times\left(\frac{1}{\int_a^b w(s)ds}\int_a^b\left|f(t)-\frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds\right|^p w(t)dt\right)^{1/p}.\end{aligned} \tag{4.11}$$

In particular, if $p=q=2$, then we have [8]*:*

$$\begin{aligned}|C_w(f,g)| &\leq \frac{2}{\pi}\left(\int_a^b w(s)ds\right)^{1/2}\left(\int_a^b \frac{|g'(t)|^2}{w(t)}dt\right)^{1/2} \\ &\times\left(\frac{1}{\int_a^b w(s)ds}\int_a^b f^2(t)w(t)dt-\left(\frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds\right)^2\right)^{1/2}.\end{aligned} \tag{4.12}$$

Proof. We use the following Sonin type identity

$$\begin{aligned}C_w(f,g) &= \frac{1}{\int_a^b w(s)ds} \\ &\times\int_a^b\left(f(t)-\frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds\right)(g(t)-g(a))\,w(t)dt,\end{aligned} \tag{4.13}$$

which can be proved directly on calculating the integral from the right hand side.

Using the weighted Hölder's integral inequality for $p, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, we have

$$\begin{aligned}
|C_w(f,g)| &\le \frac{1}{\int_a^b w(s)ds} \\
&\quad \times \int_a^b \left| f(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds \right| |g(t)-g(a)|\, w(t)dt \\
&\le \frac{1}{\int_a^b w(s)ds}\left(\int_a^b \left| f(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds \right|^p w(t)dt\right)^{1/p} \\
&\quad \times \left(\int_a^b |g(t)-g(a)|^q\, w(t)dt\right)^{1/q}.
\end{aligned} \tag{4.14}$$

Using (2.10) for $f = g - g(a)$, we have

$$\int_a^b |g(t)-g(a)|^q\, w(t)dt \le \frac{2^q}{(q-1)\pi_{q-1}^q}\left(\int_a^b w(s)ds\right)^q \int_a^b \frac{|g'(t)|^q}{w^{q-1}(t)}dt \tag{4.15}$$

which is equivalent to

$$\begin{aligned}
&\left(\int_a^b |g(t)-g(a)|^q\, w(t)dt\right)^{1/q} \\
&\qquad \le \frac{2}{(q-1)^{1/q}\pi_{q-1}}\left(\int_a^b w(s)ds\right)\left(\int_a^b \frac{|g'(t)|^q}{w^{q-1}(t)}dt\right)^{1/q}.
\end{aligned} \tag{4.16}$$

By making use of (4.14) and (4.16), we get

$$\begin{aligned}
&|C_w(f,g)| \\
&\le \frac{1}{\int_a^b w(s)ds}\left(\int_a^b \left| f(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds \right|^p w(t)dt\right)^{1/p} \\
&\quad \times \frac{2}{(q-1)^{1/q}\pi_{q-1}}\left(\int_a^b w(s)ds\right)\left(\int_a^b \frac{|g'(t)|^q}{w^{q-1}(t)}dt\right)^{1/q} \\
&= \frac{2}{(q-1)^{1/q}\pi_{q-1}}\left(\int_a^b w(s)ds\right)^{1/p} \\
&\quad \times \left(\frac{1}{\int_a^b w(s)ds}\int_a^b \left| f(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds \right|^p w(t)dt\right)^{1/p} \\
&\quad \times \left(\int_a^b \frac{|g'(t)|^q}{w^{q-1}(t)}dt\right)^{1/q},
\end{aligned} \tag{4.17}$$

hence by (4.17) we get the desired result (4.10). □

Remark 5. If we take $w \equiv 1$, then by (4.11) we get

$$
\begin{aligned}
|C(f,g)| &\le \frac{2}{(q-1)^{1/q}\pi_{q-1}}(b-a)^{1/p}\left(\int_a^b |g'(t)|^q\,dt\right)^{1/q} \\
&\quad\times\left(\frac{1}{b-a}\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)w(s)ds\right|^p dt\right)^{1/p}.
\end{aligned}
\tag{4.18}
$$

5. Reverses of Jensen's inequality. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \ge 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the *Lebesgue space*

$$
L_w(\Omega,\mu) := \Big\{ f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x)\,|f(x)|\,d\mu(x) < \infty \Big\}.
$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)d\mu(x)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S. S. Dragomir obtained in 2002 [7] the following result:

Theorem 7. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ be such that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L_w(\Omega, \mu)$, where $w \ge 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequality:*

$$
\begin{aligned}
0 &\le \int_\Omega w(\Phi\circ f)\,d\mu - \Phi\left(\int_\Omega wf\,d\mu\right) \\
&\le \int_\Omega w\left(\Phi'\circ f\right) f\,d\mu - \int_\Omega w\left(\Phi'\circ f\right)d\mu \int_\Omega wf\,d\mu.
\end{aligned}
\tag{5.1}
$$

We have the following reverse of Jensen's inequality:

Theorem 8. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M), $w : [a, b] \to (0, \infty)$ be continuous on $[a, b]$ and $f : [a, b] \to [m, M]$ be absolutely continuous so that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L_w[a, b]$. Assume that p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *If $(\Phi'' \circ f) f' \in L_q[a, b]$ and $f \in L_p[a, b]$, then*

$$0 \le \frac{1}{\int_a^b w(s)ds}\int_a^b w(t)\left(\Phi\circ f\right)(t)dt - \Phi\left(\frac{\int_a^b w(t)f(t)dt}{\int_a^b w(s)ds}\right)$$

$$\le \frac{b-a}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b \left|\left(\Phi''\circ f\right)(t)\right|^q \left|f'(t)\right|^q dt\right)^{1/q} \tag{5.2}$$

$$\times \frac{1}{\int_a^b w(s)ds}\left(\int_a^b \left|f(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds\right|^p w^p(t)dt\right)^{1/p}.$$

(ii) *If* $f' \in L_q[a,b]$ *and* $\Phi'\circ f \in L_p[a,b]$, *then*

$$0 \le \frac{1}{\int_a^b w(s)ds}\int_a^b w(t)\left(\Phi\circ f\right)(t)dt - \Phi\left(\frac{\int_a^b w(t)f(t)dt}{\int_a^b w(s)ds}\right)$$

$$\le \frac{b-a}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b \left|f'(t)\right|^q dt\right)^{1/q}\frac{1}{\int_a^b w(s)ds} \tag{5.3}$$

$$\times\left(\int_a^b \left|\left(\Phi'\circ f\right)(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b \left(\Phi'\circ f\right)(s)w(s)ds\right|^p w^p(t)dt\right)^{1/p}.$$

Proof. (i) From (5.1) we have

$$0 \le \frac{1}{\int_a^b w(s)ds}\int_a^b w(t)\left(\Phi\circ f\right)(t)dt - \Phi\left(\frac{\int_a^b w(t)f(t)dt}{\int_a^b w(s)ds}\right)$$

$$\le \frac{1}{\int_a^b w(s)ds}\int_a^b w(t)\left(\Phi'\circ f\right)(t)f(t)dt \tag{5.4}$$

$$- \frac{1}{\int_a^b w(s)ds}\int_a^b w(t)\left(\Phi'\circ f\right)(t)dt\cdot\frac{1}{\int_a^b w(s)ds}\int_a^b w(t)f(t)dt$$

$$= C_w\left(f, \Phi'\circ f\right).$$

From the inequality (4.4) we have

$$\left|C_w\left(f,\Phi'\circ f\right)\right| \le \frac{b-a}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b \left|\left(\Phi'\circ f\right)'(t)\right|^q dt\right)^{1/q}$$

$$\times\frac{1}{\int_a^b w(s)ds}\left(\int_a^b \left|f(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds\right|^p w^p(t)dx\right)^{1/p}$$

$$= \frac{b-a}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b \left|\left(\Phi''\circ f\right)(t)f'(t)\right|^q dt\right)^{1/q}$$

$$\times\frac{1}{\int_a^b w(s)ds}\left(\int_a^b \left|f(t) - \frac{1}{\int_a^b w(s)ds}\int_a^b f(s)w(s)ds\right|^p w^p(t)dx\right)^{1/p},$$

which together with (5.4) proves (5.2).

(ii) From (4.4) we also have

$$\begin{aligned}
&\left|C_w\left(f,\Phi'\circ f\right)\right| = \left|C_w\left(\Phi'\circ f, f\right)\right| \\
&\leq \frac{b-a}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b \left|f'(t)\right|^q dt\right)^{1/q}\frac{1}{\int_a^b w(s)ds} \\
&\quad\times\left(\int_a^b\left|\Phi'\circ f(t)-\frac{1}{\int_a^b w(s)ds}\int_a^b \Phi'\circ f(s)w(s)ds\right|^p w^p(t)dx\right)^{1/p},
\end{aligned}$$

which together with (5.4) proves (5.3). □

Corollary 4. *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:[a,b]\to[m,M]$ be absolutely continuous so that $\Phi\circ f$, f, $\Phi'\circ f$, $(\Phi'\circ f)\,f\in L\,[a,b]$. Assume that p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.*

(i) *If $(\Phi''\circ f)\,f'\in L_q\,[a,b]$ and $f\in L_p\,[a,b]$, then*

$$\begin{aligned}
&0\leq\frac{1}{b-a}\int_a^b\left(\Phi\circ f\right)(t)dt-\Phi\left(\frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq\frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b\left|\left(\Phi''\circ f\right)(t)\right|^q\left|f'(t)\right|^q dt\right)^{1/q} \qquad (5.5)\\
&\quad\times\left(\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)ds\right|^p dt\right)^{1/p}.
\end{aligned}$$

(ii) *If $f'\in L_q\,[a,b]$ and $\Phi'\circ f\in L_p\,[a,b]$, then*

$$\begin{aligned}
&0\leq\frac{1}{b-a}\int_a^b\left(\Phi\circ f\right)(t)dt-\Phi\left(\frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq\frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b\left|f'(t)\right|^q dt\right)^{1/q} \qquad (5.6)\\
&\quad\times\left(\frac{1}{b-a}\int_a^b\left|\left(\Phi'\circ f\right)(t)-\frac{1}{b-a}\int_a^b\left(\Phi'\circ f\right)(s)ds\right|^p dt\right)^{1/p}.
\end{aligned}$$

Corollary 5. *Let $\Phi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (a,b), $w : [a,b] \to (0,\infty)$ be continuous on $[a,b]$ so that Φ, Φ', $\Phi'\ell \in L_w[a,b]$, where $\ell(t) = t$. Assume that p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *If $\Phi'' \in L_q[a,b]$, then*

$$\begin{aligned} 0 &\le \frac{1}{\int_a^b w(s)ds} \int_a^b w(t)\Phi(t)dt - \Phi\left(\frac{\int_a^b tw(t)dt}{\int_a^b w(s)ds}\right) \\ &\le \frac{b-a}{(p-1)^{1/p}\pi_{p-1}} \left(\int_a^b \left|\Phi''(t)\right|^q dt\right)^{1/q} \\ &\quad \times \frac{1}{\int_a^b w(s)ds} \left(\int_a^b \left| t - \frac{1}{\int_a^b w(s)ds} \int_a^b sw(s)ds \right|^p w^p(t)dt\right)^{1/p}. \end{aligned} \tag{5.7}$$

(ii) *If $\Phi' \in L_p[a,b]$, then*

$$\begin{aligned} 0 &\le \frac{1}{\int_a^b w(s)ds} \int_a^b w(t)\Phi(t)dt - \Phi\left(\frac{\int_a^b tw(t)dt}{\int_a^b w(s)ds}\right) \\ &\le \frac{b-a}{(p-1)^{1/p}\pi_{p-1}} \frac{1}{\int_a^b w(s)ds} \\ &\quad \times \left(\int_a^b \left|\Phi'(t) - \frac{1}{\int_a^b w(s)ds} \int_a^b \Phi'(s)w(s)ds \right|^p w^p(t)dt\right)^{1/p}. \end{aligned} \tag{5.8}$$

We also have:

Theorem 9. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m,M), $w : [a,b] \to (0,\infty)$ be continuous on $[a,b]$ and $f : [a,b] \to [m,M]$ be absolutely continuous so that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L_w[a,b]$. Assume that p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *If $|\Phi'' \circ f|^q |f'|^q w^{1-q} \in L[a,b]$ and $f \in L_p[a,b]$, then*

$$\begin{aligned} 0 &\le \frac{1}{\int_a^b w(s)ds} \int_a^b w(t)(\Phi \circ f)(t)dt - \Phi\left(\frac{\int_a^b w(t)f(t)dt}{\int_a^b w(s)ds}\right) \\ &\le \frac{2}{(q-1)^{1/q}\pi_{q-1}} \left(\int_a^b w(s)ds\right)^{1/p} \left(\int_a^b \frac{|(\Phi'' \circ f)(t)|^q |f'(t)|^q}{w^{q-1}(t)} dt\right)^{1/q} \\ &\quad \times \left(\frac{1}{\int_a^b w(s)ds} \int_a^b \left| f(t) - \frac{1}{\int_a^b w(s)ds} \int_a^b f(s)w(s)ds \right|^p w(t)dt\right)^{1/p}. \end{aligned} \tag{5.9}$$

(ii) *If* $|f'|^q w^{1-q} \in L[a,b]$ *and* $\Phi' \circ f \in L_p[a,b]$, *then*

$$0 \le \frac{1}{\int_a^b w(s)ds} \int_a^b w(t)(\Phi \circ f)(t)dt - \Phi\left(\frac{\int_a^b w(t)f(t)dt}{\int_a^b w(s)ds}\right)$$
$$\le \frac{2}{(q-1)^{1/q}\pi_{q-1}} \left(\int_a^b w(s)ds\right)^{1/p} \left(\int_a^b \frac{|f'(t)|^q}{w^{q-1}(t)}dt\right)^{1/q}$$
$$\times \left(\frac{1}{\int_a^b w(s)ds} \int_a^b \left|(\Phi' \circ f)(t) - \frac{1}{\int_a^b w(s)ds} \int_a^b (\Phi' \circ f)(s)w(s)ds\right|^p w(t)dt\right)^{1/p}.$$

The proof follows in a similar way by utilising the inequality (4.11). The details are not provided here.

Corollary 6. *Let* $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable convex function on* (m,M) *and* $f : [a,b] \to [m,M]$ *be absolutely continuous so that* $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L[a,b]$. *Assume that* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

(i) *If* $(\Phi'' \circ f) f' \in L_q[a,b]$ *and* $f \in L_p[a,b]$, *then*

$$0 \le \frac{1}{b-a} \int_a^b (\Phi \circ f)(t)dt - \Phi\left(\frac{1}{b-a} \int_a^b f(t)dt\right)$$
$$\le \frac{2}{(q-1)^{1/q}\pi_{q-1}}(b-a)^{1/p} \left(\int_a^b \left|(\Phi'' \circ f)(t)\right|^q \left|f'(t)\right|^q dt\right)^{1/q}$$
$$\times \left(\frac{1}{b-a} \int_a^b \left|f(t) - \frac{1}{b-a} \int_a^b f(s)ds\right|^p dt\right)^{1/p}.$$

(ii) *If* $f' \in L_q[a,b]$ *and* $\Phi' \circ f \in L_p[a,b]$, *then*

$$0 \le \frac{1}{b-a} \int_a^b (\Phi \circ f)(t)dt - \Phi\left(\frac{1}{b-a} \int_a^b f(t)dt\right)$$
$$\le \frac{2}{(q-1)^{1/q}\pi_{q-1}}(b-a)^{1/p} \left(\int_a^b \left|f'(t)\right|^q dt\right)^{1/q}$$
$$\times \left(\frac{1}{b-a} \int_a^b \left|(\Phi' \circ f)(t) - \frac{1}{b-a} \int_a^b (\Phi' \circ f)(s)ds\right|^p dt\right)^{1/p}.$$

Finally, we have:

Corollary 7. *Let* $\Phi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable convex function on* (a,b), $w : [a,b] \to (0,\infty)$ *be continuous on* $[a,b]$ *so that* Φ, Φ', $\Phi'\ell \in L_w[a,b]$, *where* $\ell(t) = t$. *Assume that* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

(i) *If* $|\Phi''|^q w^{1-q} \in L[a,b]$, *then*

$$0 \le \frac{1}{\int_a^b w(s)ds} \int_a^b w(t)\Phi(t)dt - \Phi\left(\frac{\int_a^b tw(t)dt}{\int_a^b w(s)ds}\right)$$
$$\le \frac{2}{(q-1)^{1/q}\pi_{q-1}} \left(\int_a^b w(s)ds\right)^{1/p} \left(\int_a^b \frac{|\Phi''(t)|^q}{w^{q-1}(t)}dt\right)^{1/q}$$
$$\times \left(\frac{1}{\int_a^b w(s)ds} \int_a^b \left|t - \frac{1}{\int_a^b w(s)ds} \int_a^b f(s)sds\right|^p w(t)dt\right)^{1/p}.$$

(ii) *If* $w^{1-q} \in L[a,b]$ *and* $\Phi' \in L_p[a,b]$, *then*

$$0 \le \frac{1}{\int_a^b w(s)ds} \int_a^b w(t)\Phi(t)dt - \Phi\left(\frac{\int_a^b tw(t)dt}{\int_a^b w(s)ds}\right)$$
$$\le \frac{2}{(q-1)^{1/q}\pi_{q-1}} \left(\int_a^b w(s)ds\right)^{1/p} \left(\int_a^b \frac{1}{w^{q-1}(t)}dt\right)^{1/q}$$
$$\times \left(\frac{1}{\int_a^b w(s)ds} \int_a^b \left|\Phi'(t) - \frac{1}{\int_a^b w(s)ds} \int_a^b \Phi'(s)w(s)ds\right|^p w(t)dt\right)^{1/p}.$$

Acknowledgement. The author would like to thank the anonymous referee for valuable suggestions that have been implemented in the final version of the manuscript.

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Received May 3, 2021