doi: 10.17951/a.2022.76.1.25-30

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXVI, NO. 1, 2022 SECTIO A 25–30

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A note on the Banach–Mazur distances between c_0 and other ℓ_1-preduals

ABSTRACT. We prove that if X is an ℓ_1-predual isomorphic to the space c_0 of sequences converging to zero, then for any isomorphism $T : X \to c_0$ we have $\|T\| \, \|T^{-1}\| \geq 1+2r^*(X)$, where $r^*(X)$ is the smallest radius of the closed ball of the dual space X^* containing all the weak* cluster points of the set of all extreme points of the closed unit ball of X^*.

1. Introduction. Let X be a real infinite-dimensional Banach space X and let us denote by B_X its closed unit ball. If $A \subset X$, then $\operatorname{ext} A$ stands for the set of all extreme points of A. The dual of X is denoted by X^*. If $A \subset X^*$, then $\overline{A}^*$ denotes the weak* closure of A and A' stands for the set of all weak* cluster points of A:

$$A' = \left\{ x^* \in X^* : x^* \in \overline{(A \setminus \{x^*\})}^* \right\}.$$

If $f \in X^*$, then $\ker f$ denotes the kernel of f, i.e., $\ker f = \{x \in X : f(x) = 0\}$. For any Banach spaces X and Y, $X = Y$ means that X is isometrically isomorphic to Y. A Banach space X is called an L_1-predual (or a Lindenstrauss space) if $X^* = L_1(\mu)$ for some measure μ. In particular, X is named an ℓ_1-predual if $X^* = \ell_1$. For a given ℓ_1-predual X we put

$$r^*(X) = \inf\{r > 0 : (\operatorname{ext} B_{X^*})' \subset rB_{X^*}\} = \sup\{\|e^*\| : e^* \in (\operatorname{ext} B_{X^*})'\}.$$

2010 *Mathematics Subject Classification.* 46B03, 46B25, 46B45.
Key words and phrases. ℓ_1-preduals, Banach–Mazur distance, c_0 space.

For Banach spaces X and Y, a linear operator $T : X \to Y$ is called an isomorphic embedding if there exist $a, b > 0$ such that for every $x \in X$

$$a \|x\| \leq \|T(x)\| \leq b \|x\| .$$

The distortion of an isomorphic embedding $T : X \to Y$ is the number $\|T\| \|T^{-1}\|$, where T^{-1} denotes the inverse map to an isomorphism T of X onto its image $T(X)$. Moreover, for isomorphic Banach spaces X and Y, $d(X, Y)$ denotes the Banach–Mazur distance between them, defined as

$$d(X, Y) = \inf \left\{ \|T\| \|T^{-1}\| : T \text{ is an isomorphism from } X \text{ onto } Y \right\} .$$

This notion appeared for the first time in the celebrated 1932' book by Stefan Banach [3]. The reader interested in the current state of knowledge regarding the Banach–Mazur distance between L_1-preduals is referred to the paper [8] and the papers cited in it. One of the most important classical result is the Cambern result [4], which states that the Banach–Mazur distance between the space c of convergent sequences and its subspace c_0 of sequences converging to zero equals 3, both spaces are furnished with the supremum norm. This result answered to the question posed by Banach in [3]. In the present paper, we prove that the Banach–Mazur distance between c_0 and an ℓ_1-predual X isomorphic to c_0 is greater or equal to $1 + 2r^*(X)$. It is worth emphasizing that this estimate is optimal (see Remark 2.8). This result is a generalization of Theorem 3.7 in [6], where some ℓ_1-preduals X isomorphic to c_0, for which $r^*(X) = 1$, are considered. Moreover, this result complements Theorem 2.1 in [8] and Theorem 4.1 in [8].

We recall that c^* can be isometrically identified with ℓ_1 in the following way. For every $x^* \in c^*$ there exists a unique $f = (f(1), f(2), \dots) \in \ell_1$ such that

$$x^*(x) = \sum_{i=0}^{\infty} f(i+1) x(i) = f(x)$$

with $x = (x(1), x(2), \dots) \in c$ and $x(0) = \lim_{i \to \infty} x(i)$. In our paper, ℓ_1-predual hyperplanes in c play an important role.

For every $e^* = (e^*(1), e^*(2), \dots) \in \ell_1$ we define a hyperplane W_{e^*} in c by

$$W_{e^*} = \left\{ x = (x(1), x(2), \dots) \in c : \lim_{i \to \infty} x(i) = \sum_{i=1}^{\infty} e^*(i) x(i) \right\} .$$

Theorem 1.1 ([5]).

(*i*) *$W_{e^*}^* = \ell_1$ if and only if one of the following conditions holds:*
- *$e^* \in B_{\ell_1}$,*
- *$\|e^*\| > 1$ and $|e^*(i)| \geq \frac{1}{2}(1 + \|e^*\|)$ for some $i \in \mathbb{N}$ (in this case, $W_{e^*} = c$).*

(*ii*) *Let $e^* \in B_{\ell_1}$. Then $W_{e^*} = c$ if and only if $|e^*(i)| = 1$ for some $i \in \mathbb{N}$. Moreover, $W_{e^*} = c_0$ if and only if $e^* = (0, 0, 0, \dots)$.*

(iii) *For every* $e^* \in B_{\ell_1}$ *we have* $W_{e^*}^* = \ell_1$ *with a duality map* $\phi : \ell_1 \to W_{e^*}^*$ *defined by*

$$\phi(g)(x) = \sum_{i=1}^{\infty} x(i)g(i)$$

with $g = (g(1), g(2), \dots) \in \ell_1$ *and* $x = (x(1), x(2), \dots) \in W_{e^*}$*. Moreover, if* (e_n^*) *denotes the standard basis in* ℓ_1*, then*

$$e_n^* \xrightarrow{\sigma(\ell_1, W_{e^*})} e^*,$$

where $\sigma(X^*, X)$ *denotes the weak* topology on* X^* *induced by* X*.*

(iv) *If* X *is an* ℓ_1*-predual such that* (e_n^*) *is* $\sigma(\ell_1, X)$*-convergent to* e^**, then* $X = W_{e^*}$*.*

Note that in the present paper we use a slight modification of the notation for a hyperplane in c introduced in [5]. Indeed, here we have

$$W_{e^*} = W_f = \ker f = \left\{ x \in c : f(1) \lim_{i\to\infty} x(i) + \sum_{i=1}^{\infty} f(i+1)x(i) = 0 \right\},$$

where

$$f = \left(\frac{1}{1+\|e^*\|}, -\frac{e^*(1)}{1+\|e^*\|}, -\frac{e^*(2)}{1+\|e^*\|}, \dots, -\frac{e^*(i)}{1+\|e^*\|}, \dots \right) \in S_{c^*}.$$

2. Main result. We begin by stating the main result of the paper.

Theorem 2.1. *If* X *is an* ℓ_1*-predual isomorphic to* c_0*, then*

$$d(X, c_0) \geq 1 + 2r^*(X).$$

In order to prove the theorem we need some auxiliary results.

Theorem 2.2 (see, e.g., [10]). *Let* $T : X \to Y$ *be a bounded linear map from a Banach space* X *onto a Banach space* Y*. Then there exists a linear map* $\widetilde{T} : X/\ker T \to Y$ *such that*

1) $\widetilde{T}$ *is isomorphism,*
2) $T = \widetilde{T}\pi$*, where* $\pi : X \to X/\ker T$ *denotes the quotient map and* $\ker T = \{x \in X : T(x) = 0\}$*,*
3) $\|T\| = \|\widetilde{T}\|$.

Theorem 2.3 ([1]). *Let* X *be a quotient of* c_0*. Then for every* $\varepsilon > 0$*, there is a subspace* Y *of* c_0 *such that* $d(X, Y) < 1 + \varepsilon$*.*

Lemma 2.4 (Lemma 1 in [2]). *Let* X *be a Banach space with separable dual* X^* *and let* Y *be a subspace of* X^* *with a normalized basis* (y_n^*) *which is isomorphic to* ℓ_1*. If* $\overline{\{y_n^* : n \in \mathbb{N}\}}^* \subset Y$*, then* Y *is weak* closed in* X^**.*

Lemma 2.5 (Lemma 2 in [2]). *Suppose that X and Y are separable Banach spaces and that (x_n^*) and (y_n^*) are normalized sequences in X^* and Y^*, respectively, which are equivalent to the standard basis of ℓ_1 and for which $\overline{\overline{\operatorname{lin}\{x_n^* : n \in \mathbb{N}\}}}^* = \overline{\operatorname{lin}\{x_n^* : n \in \mathbb{N}\}}$ and $\overline{\overline{\operatorname{lin}\{y_n^* : n \in \mathbb{N}\}}}^* = \overline{\operatorname{lin}\{y_n^* : n \in \mathbb{N}\}}$. Suppose that the basis to basis map ϕ of $\overline{\operatorname{lin}\{x_n^* : n \in \mathbb{N}\}}$ onto $\overline{\operatorname{lin}\{y_n^* : n \in \mathbb{N}\}}$, i.e.,*

$$\phi\left(\sum_{n=1}^{\infty} a_n x_n^*\right) = \sum_{n=1}^{\infty} a_n y_n^*$$

is a weak* homeomorphism of $\overline{\{x_n^* : n \in \mathbb{N}\}}^*$ onto $\overline{\{y_n^* : n \in \mathbb{N}\}}^*$. Then ϕ is a weak* continuous isomorphism of $\overline{\operatorname{lin}\{x_n^* : n \in \mathbb{N}\}}$ onto $\overline{\operatorname{lin}\{y_n^* : n \in \mathbb{N}\}}$.

Lemma 2.6 (Lemma 3.2 in [6]). *Let $T : X \to Y$ be a bounded linear operator, where $Y \neq \{0\}$. Then*

$$\sup\{\delta > 0 : \delta B_Y \subseteq T(B_X)\} = \left\|\widetilde{T}^{-1}\right\|^{-1},$$

where $\widetilde{T}$ is defined as in Theorem 2.2.

Theorem 2.7 (Theorem 4.1 in [8]). *Let $e^* \in B_{\ell_1}$ and let X be an infinite-dimensional L_1-predual such that $(\operatorname{ext} B_{X^*})' \subset rB_{X^*}$ for some $0 \leq r < \|e^*\|$. Then for every isomorphic embedding T from W_{e^*} into X we have*

$$\|T\| \, \|T^{-1}\| \geq \frac{1 + 2\,\|e^*\| - r}{1 + r}.$$

We are now in position to prove the main theorem of this paper.

Proof of Theorem 2.1. Observe that, if $r^*(X) = 0$, then $X = c_0$ (see [7]). Therefore, assume that $r^*(X) > 0$. Let $\varepsilon \in (0, r^*(X))$ be arbitrarily chosen. There exist $e^* \in (\operatorname{ext} B_{X^*})'$ and a subsequence $(e^*_{n_k})_{k\in\mathbb{N}}$ of the standard basis in ℓ_1 such that $\|e^*\| > r^*(X) - \frac{\varepsilon}{2}$, $e^*_{n_k} \xrightarrow{\sigma(\ell_1, X)} e^*$ and $\|e^*\| > \sum_{k=1}^{\infty} |e^*(n_k)|$. Put

$$e^*_{n_0} = \frac{e^* - \sum_{k=1}^{\infty} e^*(n_k) e^*_{n_k}}{\|e^*\| - \sum_{k=1}^{\infty} |e^*(n_k)|}.$$

It is easy to see that $\|e^*_{n_0}\| = 1$ and the sequence $(e^*_{n_k})_{k\in\mathbb{N}\cup\{0\}}$ is equivalent to the standard basis in ℓ_1. Let $Y = \overline{\operatorname{lin}\{e^*_{n_0}, e^*_{n_1}, e^*_{n_2}, \dots\}}$. Since $\overline{\{e^*_{n_0}, e^*_{n_1}, e^*_{n_2}, \dots\}}^* = \{e^*_{n_0}, e^*_{n_1}, e^*_{n_2}, \dots\} \cup \{e^*\} \subset Y$, Lemma 2.4 guarantees that $\overline{Y}^* = Y$. Thus $Y = (X/{}^{\perp}Y)^*$. Let

$$y^* = \left(\|e^*\| - \sum_{k=1}^{\infty} |e^*(n_k)|\,, e^*(n_1), e^*(n_2), e^*(n_3), \dots\right).$$

Since $y^* \in B_{\ell_1}$, by Theorem 1.1, $W^*_{y^*} = \ell_1$ and $e^*_n \xrightarrow{\sigma(\ell_1, W_{y^*})} y^*$. Let $\phi : Y \to W^*_{y^*}$ be defined as follows:

$$\phi(a_1 e^*_{n_0} + a_2 e^*_{n_1} + a_3 e^*_{n_2} + a_4 e^*_{n_3} + \dots) = \sum_{k=1}^{\infty} a_k e^*_k.$$

Then ϕ is an "onto" linear isometry. Moreover,

$$\begin{aligned}\phi(e^*) &= \phi\left(\left(\|e^*\| - \sum_{k=1}^{\infty} |e^*(n_k)|\right) e^*_{n_0} + \sum_{k=1}^{\infty} e^*(n_k) e^*_{n_k}\right) \\ &= \left(\|e^*\| - \sum_{k=1}^{\infty} |e^*(n_k)|\right) e^*_1 + \sum_{k=1}^{\infty} e^*(n_k) e^*_{k+1} \\ &= \left(\|e^*\| - \sum_{k=1}^{\infty} |e^*(n_k)|, e^*(n_1), e^*(n_2), e^*(n_3), \dots\right) = y^*.\end{aligned}$$

Consequently, ϕ is a weak* continuous homeomorphism from

$$\overline{\{e^*_{n_0}, e^*_{n_1}, e^*_{n_2}, \dots\}}^* = \{e^*_{n_0}, e^*_{n_1}, e^*_{n_2}, \dots\} \cup \{e^*\}$$

onto

$$\overline{\{e^*_1, e^*_2, \dots\}}^* = \{e^*_1, e^*_2, \dots\} \cup \{y^*\}.$$

In view of Lemma 2.5, ϕ is a weak* continuous isometry from Y onto $\ell_1 = W^*_{y^*}$. This implies that W_{y^*} is isometric to $X/^{\perp}Y$.

Now, assume that $T : X \to c_0$ is an isomorphism. Without loss of generality we may assume that $\|T^{-1}\| = 1$. Let us consider the map $\pi T^{-1} : c_0 \to X/^{\perp}Y = W_{y^*}$, where $\pi : X \to X/^{\perp}Y$ is the quotient map. Obviously πT^{-1} is an "onto" map. By Theorem 2.2, there exists an isomorphism $\widetilde{\pi T^{-1}} : c_0/\ker \pi T^{-1} \to W_{y^*}$ such that $\left\|\widetilde{\pi T^{-1}}\right\| = \|\pi T^{-1}\|$. Observe that $\pi T^{-1}(B_{c_0}) \supseteq \frac{1}{\|T\|+\eta} B_{W_{y^*}}$ for every $\eta > 0$. Hence, by applying Lemma 2.6, we obtain $\|T\| \geq \left\|\left(\widetilde{\pi T^{-1}}\right)^{-1}\right\|$. Since $\|\pi T^{-1}\| \leq 1$, we have $\left\|\widetilde{\pi T^{-1}}\right\| \leq 1$.

Now observe that, by Theorem 2.3, there exist a subspace Z of c_0 and an isomorphism $K : c_0/\ker \pi T^{-1} \to Z$ such that $\|K\|\,\|K^{-1}\| < 1 + \varepsilon$. Hence, applying Theorem 4.1 in [9], we obtain

$$\begin{aligned}1 + 2\|y^*\| &\leq \left\|\widetilde{\pi T^{-1}} K^{-1}\right\| \left\|K \left(\widetilde{\pi T^{-1}}\right)^{-1}\right\| \\ &\leq \|K^{-1}\| \left\|\widetilde{\pi T^{-1}}\right\| \|K\| \left\|\left(\widetilde{\pi T^{-1}}\right)^{-1}\right\| \leq (1+\varepsilon)\|T\|.\end{aligned}$$

Therefore $\|T\| \geq \frac{1+2\|e^*\|}{1+\varepsilon} > \frac{1+2r^*(X)-\varepsilon}{1+\varepsilon}$. Letting $\varepsilon \to 0$, we get

$$\|T\|\,\|T^{-1}\| \geq 1 + 2r^*(X).$$

□

Remark 2.8. From the proof of Proposition 3.8 in [6] we have $d(W_{e^*}, c_0) \leq 1 + 2\|e^*\|$. Applying Theorem 2.1 or Theorem 2.7, we conclude that $d(W_{e^*}, c_0) = 1 + 2\|e^*\|$ for every $e^* \in B_{\ell_1}$.

Acknowledgments. The author would like to thank dr hab. Łukasz Piasecki for helpful conversations and valuable suggestions.

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Received July 7, 2022