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Matrix representations of third order jet groups

ABSTRACT. In this paper, faithful matrix representations of the jet groups G^3_n are presented, following a detailed description of their components in block form. Such groups can be used further to study symmetries of differential equations. Elements of these matrix representations are derived.

1. Introduction. Jet groups play an important role in differential geometry and modern physics. They can be used as a tool to study the symmetries of differential equations [4]. Nevertheless, for practical applications, it is necessary to know the matrix representations of these jet groups.

Consider the n-dimensional manifold M and the rth order frame bundle P^rM of M, which is the set of all r-jets with source $\mathbf{0}$ of the local diffeomorphisms of $\mathbb{R}^n$ into M. P^rM is an open subset of $T^r_nM := J^r_{\mathbf{0}}(\mathbb{R}^n, M)$ and therefore defines the structure of a smooth fiber bundle $P^rM \to M$. The jet group G^r_n acts smoothly on P^rM on the right by jet composition, and thus P^rM becomes a principle bundle with the structure group G^r_n. If we construct a matrix representation of G^r_n structure group, we can also represent this right action by matrix multiplication, which is a much easier tool to work with and is useful in practical calculations. For more detailed explanation and further information on jet groups, see [5] and [2].

Since results and matrix representations are known for jet groups of order at most $r = 2$, our attention in this paper will be devoted to jet groups of order $r = 3$. Jet group G^3_3 will be presented, and elements of this representation will be derived. As a corollary, faithful representations of G^3_2, G^3_1 and G^3_n, $n \in \mathbb{N}$, jet groups will also be presented (the case of G^3_n is also a corollary, since the proof is completely analogous to the one in the case of

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G_3^3). Detailed derivation of matrix representations of jet groups G_1^r, G_n^1, G_2^2 and G_3^2 for $r, n \in \mathbb{N}$ can be found in [1] and [3].

2. Matrix representation of G_3^3. First, let us show what a faithful matrix representation of the jet group G_3^3 looks like. For a particular element of this matrix, the upper indices will denote the component of a map we differentiate, and the lower indices will denote the coordinate by which we differentiate. For example, element a^2_{113} for a map $f : \mathbb{R}^3 \to \mathbb{R}^3$ with components $f(x, y, z) = (f^1(x, y, z), f^2(x, y, z), f^3(x, y, z))$ corresponds to a third derivation of a second component calculated at the origin as

$$\frac{\partial^3 f^2}{\partial x^2 \partial z}(\mathbf{0}).$$

The number of lower indices thus represents the order of a derivation.

Theorem 1. *The faithful matrix representation of the jet group G_3^3 has the form*

$$\alpha = \begin{pmatrix} A & B & C \\ O & D & E \\ O & O & F \end{pmatrix} \in M_{19}(\mathbb{R}),$$

where O are zero matrices and

$$A = \begin{pmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{pmatrix}, \quad B = \begin{pmatrix} a^1_{11} & a^1_{22} & a^1_{33} & a^1_{12} & a^1_{13} & a^1_{23} \\ a^2_{11} & a^2_{22} & a^2_{33} & a^2_{12} & a^2_{13} & a^2_{23} \\ a^3_{11} & a^3_{22} & a^3_{33} & a^3_{12} & a^3_{13} & a^3_{23} \end{pmatrix},$$

$$C = \begin{pmatrix} a^1_{111} & a^1_{222} & a^1_{333} & a^1_{123} & a^1_{112} & a^1_{113} & a^1_{122} & a^1_{133} & a^1_{223} & a^1_{233} \\ a^2_{111} & a^2_{222} & a^2_{333} & a^2_{123} & a^2_{112} & a^2_{113} & a^2_{122} & a^2_{133} & a^2_{223} & a^2_{233} \\ a^3_{111} & a^3_{222} & a^3_{333} & a^3_{123} & a^3_{112} & a^3_{113} & a^3_{122} & a^3_{133} & a^3_{223} & a^3_{233} \end{pmatrix},$$

$$D = \begin{pmatrix} (a_1^1)^2 & (a_2^1)^2 & (a_3^1)^2 & a_1^1 a_2^1 & a_1^1 a_3^1 & a_2^1 a_3^1 \\ (a_1^2)^2 & (a_2^2)^2 & (a_3^2)^2 & a_1^2 a_2^2 & a_1^2 a_3^2 & a_2^2 a_3^2 \\ (a_1^3)^2 & (a_2^3)^2 & (a_3^3)^2 & a_1^3 a_2^3 & a_1^3 a_3^3 & a_2^3 a_3^3 \\ 2a_1^1 a_1^2 & 2a_2^1 a_2^2 & 2a_3^1 a_3^2 & a_1^1 a_2^2 + a_2^1 a_1^2 & a_1^1 a_3^2 + a_3^1 a_1^2 & a_2^1 a_3^2 + a_3^1 a_2^2 \\ 2a_1^1 a_1^3 & 2a_2^1 a_2^3 & 2a_3^1 a_3^3 & a_1^1 a_2^3 + a_2^1 a_1^3 & a_1^1 a_3^3 + a_3^1 a_1^3 & a_2^1 a_3^3 + a_3^1 a_2^3 \\ 2a_1^2 a_1^3 & 2a_2^2 a_2^3 & 2a_3^2 a_3^3 & a_1^2 a_2^3 + a_2^2 a_1^3 & a_1^2 a_3^3 + a_3^2 a_1^3 & a_2^2 a_3^3 + a_3^2 a_2^3 \end{pmatrix},$$

$$E = \begin{pmatrix} a_i^1 a_{jk}^1 + a_j^1 a_{ik}^1 + a_k^1 a_{ij}^1 \\ a_i^2 a_{jk}^2 + a_j^2 a_{ik}^2 + a_k^2 a_{ij}^2 \\ a_i^3 a_{jk}^3 + a_j^3 a_{ik}^3 + a_k^3 a_{ij}^3 \\ a_i^1 a_{jk}^2 + a_j^1 a_{ik}^2 + a_k^1 a_{ij}^2 + a_i^2 a_{jk}^1 + a_j^2 a_{ik}^1 + a_k^2 a_{ij}^1 \\ a_i^1 a_{jk}^3 + a_j^1 a_{ik}^3 + a_k^1 a_{ij}^3 + a_i^3 a_{jk}^1 + a_j^3 a_{ik}^1 + a_k^3 a_{ij}^1 \\ a_i^2 a_{jk}^3 + a_j^2 a_{ik}^3 + a_k^2 a_{ij}^3 + a_i^3 a_{jk}^2 + a_j^3 a_{ik}^2 + a_k^3 a_{ij}^2 \end{pmatrix},$$

$$F = \begin{pmatrix} a_i^1 a_j^1 a_k^1 \\ a_i^2 a_j^2 a_k^2 \\ a_i^3 a_j^3 a_k^3 \\ a_i^1 a_j^2 a_k^3 + a_i^1 a_k^2 a_j^3 + a_j^1 a_i^2 a_k^3 + a_j^1 a_k^2 a_i^3 + a_k^1 a_i^2 a_j^3 + a_k^1 a_j^2 a_i^3 \\ a_i^1 a_j^1 a_k^2 + a_j^1 a_k^1 a_i^2 + a_i^1 a_k^1 a_j^2 \\ a_i^1 a_j^1 a_k^3 + a_j^1 a_k^1 a_i^3 + a_i^1 a_k^1 a_j^3 \\ a_i^1 a_j^2 a_k^2 + a_j^1 a_k^2 a_i^2 + a_i^1 a_k^2 a_j^2 \\ a_i^1 a_j^3 a_k^3 + a_j^1 a_k^3 a_i^3 + a_i^1 a_k^3 a_j^3 \\ a_i^2 a_j^2 a_k^3 + a_j^2 a_k^2 a_i^3 + a_i^2 a_k^2 a_j^3 \\ a_i^2 a_j^3 a_k^3 + a_j^2 a_k^3 a_i^3 + a_i^2 a_k^3 a_j^3 \end{pmatrix},$$

where indices ijk in matrices E and F are given respectively for each column by the following table:

Column	*1.*	*2.*	*3.*	*4.*	*5.*	*6.*	*7.*	*8.*	*9.*	*10.*
Indices ijk	*111*	*222*	*333*	*123*	*112*	*113*	*122*	*133*	*223*	*233*

Remark 1. Index notation was introduced for matrices E and F because both are too large (the matrix E has 6 rows, 10 columns and the matrix F has 10 rows, 10 columns). To make this notation clear, we can show examples. The first, fourth, and tenth columns of the matrix E have the form

$$E_1 = \begin{pmatrix} 3a_1^1 a_{11}^1 \\ 3a_1^2 a_{11}^2 \\ 3a_1^3 a_{11}^3 \\ 3(a_{11}^1 a_1^2 + a_1^1 a_{11}^2) \\ 3(a_{11}^1 a_1^3 + a_1^1 a_{11}^3) \\ 3(a_{11}^2 a_1^3 + a_1^2 a_{11}^3) \end{pmatrix},$$

$$E_4 = \begin{pmatrix} a_1^1 a_{23}^1 + a_2^1 a_{13}^1 + a_3^1 a_{12}^1 \\ a_1^2 a_{23}^2 + a_2^2 a_{13}^2 + a_3^2 a_{12}^2 \\ a_1^3 a_{23}^3 + a_2^3 a_{13}^3 + a_3^3 a_{12}^3 \\ a_1^1 a_{23}^2 + a_2^1 a_{13}^2 + a_3^1 a_{12}^2 + a_1^2 a_{23}^1 + a_2^2 a_{13}^1 + a_3^2 a_{12}^1 \\ a_1^1 a_{23}^3 + a_2^1 a_{13}^3 + a_3^1 a_{12}^3 + a_1^3 a_{23}^1 + a_2^3 a_{13}^1 + a_3^3 a_{12}^1 \\ a_1^2 a_{23}^3 + a_2^2 a_{13}^3 + a_3^2 a_{12}^3 + a_1^3 a_{23}^2 + a_2^3 a_{13}^2 + a_3^3 a_{12}^2 \end{pmatrix},$$

$$E_{10} = \begin{pmatrix} a_2^1 a_{33}^1 + 2a_3^1 a_{23}^1 \\ a_2^2 a_{33}^2 + 2a_3^2 a_{23}^2 \\ a_2^3 a_{33}^3 + 2a_3^3 a_{23}^3 \\ a_2^1 a_{33}^2 + 2a_3^1 a_{23}^2 + 2a_2^2 a_{33}^1 + a_3^2 a_{23}^1 \\ a_2^1 a_{33}^3 + 2a_3^1 a_{23}^3 + 2a_2^3 a_{33}^1 + a_3^3 a_{23}^1 \\ a_2^2 a_{33}^3 + 2a_3^2 a_{23}^3 + 2a_2^3 a_{33}^2 + a_3^3 a_{23}^2 \end{pmatrix}.$$

The index notation works in the same fashion for a matrix F.

Proof. Matrices A, B, and D of the representation matrix α form the group G_3^2, whose elements were derived and can be found in [1]. Let us derive the fourth column of matrices C, E and F, from which it is then easy to see the pattern for the rest of the columns.

Consider maps $f:\mathbb{R}^3\to\mathbb{R}^3$, $f(\mathbf{0})=\mathbf{0}$, $g:\mathbb{R}^3\to\mathbb{R}^3$, $g(\mathbf{0})=\mathbf{0}$ and the composition map $h=f\circ g=(f^i(g^1(x,y,z),g^2(x,y,z),g^3(x,y,z)))$ for $i=1,2,3$. To derive the fourth column of matrices C, E and F, we need to calculate the partial derivative

$$\frac{\partial^3 h^i}{\partial x\partial y\partial z}(\mathbf{0}).$$

Starting with the derivative $\frac{\partial h^i}{\partial x}$, we get

$$\frac{\partial h^i}{\partial x}=\frac{\partial f^i}{\partial x}\frac{\partial g^1}{\partial x}+\frac{\partial f^i}{\partial y}\frac{\partial g^2}{\partial x}+\frac{\partial f^i}{\partial z}\frac{\partial g^3}{\partial x},$$

furthermore

$$\begin{aligned}\frac{\partial^2 h^i}{\partial x\partial y}&=\left(\frac{\partial^2 f^i}{\partial x^2}\frac{\partial g^1}{\partial y}+\frac{\partial^2 f^i}{\partial x\partial y}\frac{\partial g^2}{\partial y}+\frac{\partial^2 f^i}{\partial x\partial z}\frac{\partial g^3}{\partial y}\right)\frac{\partial g^1}{\partial x}+\frac{\partial f^i}{\partial x}\frac{\partial^2 g^1}{\partial x\partial y}\\&\quad+\left(\frac{\partial^2 f^i}{\partial x\partial y}\frac{\partial g^1}{\partial y}+\frac{\partial^2 f^i}{\partial y^2}\frac{\partial g^2}{\partial y}+\frac{\partial^2 f^i}{\partial y\partial z}\frac{\partial g^3}{\partial y}\right)\frac{\partial g^2}{\partial x}+\frac{\partial f^i}{\partial y}\frac{\partial^2 g^2}{\partial x\partial y}\\&\quad+\left(\frac{\partial^2 f^i}{\partial x\partial z}\frac{\partial g^1}{\partial y}+\frac{\partial^2 f^i}{\partial y\partial z}\frac{\partial g^2}{\partial y}+\frac{\partial^2 f^i}{\partial z^2}\frac{\partial g^3}{\partial y}\right)\frac{\partial g^3}{\partial x}+\frac{\partial f^i}{\partial z}\frac{\partial^2 g^3}{\partial x\partial y}\\&=\frac{\partial^2 f^i}{\partial x^2}\frac{\partial g^1}{\partial x}\frac{\partial g^1}{\partial y}+\frac{\partial^2 f^i}{\partial y^2}\frac{\partial g^2}{\partial x}\frac{\partial g^2}{\partial y}+\frac{\partial^2 f^i}{\partial z^2}\frac{\partial g^3}{\partial x}\frac{\partial g^3}{\partial y}\\&\quad+\frac{\partial^2 f^i}{\partial x\partial y}\left(\frac{\partial g^1}{\partial x}\frac{\partial g^2}{\partial y}+\frac{\partial g^1}{\partial y}\frac{\partial g^2}{\partial x}\right)+\frac{\partial^2 f^i}{\partial x\partial z}\left(\frac{\partial g^1}{\partial x}\frac{\partial g^3}{\partial y}+\frac{\partial g^1}{\partial y}\frac{\partial g^3}{\partial x}\right)\\&\quad+\frac{\partial^2 f^i}{\partial y\partial z}\left(\frac{\partial g^2}{\partial x}\frac{\partial g^3}{\partial y}+\frac{\partial g^2}{\partial y}\frac{\partial g^3}{\partial x}\right)+\frac{\partial f^i}{\partial x}\frac{\partial^2 g^1}{\partial x\partial y}\\&\quad+\frac{\partial f^i}{\partial y}\frac{\partial^2 g^2}{\partial x\partial y}+\frac{\partial f^i}{\partial z}\frac{\partial^2 g^3}{\partial x\partial y}.\end{aligned}$$

Now, we can finally compute

$$\begin{aligned}\frac{\partial^3 h^i}{\partial x\partial y\partial z}&=\left(\frac{\partial^3 f^i}{\partial x^3}\frac{\partial g^1}{\partial z}+\frac{\partial^3 f^i}{\partial x^2\partial y}\frac{\partial g^2}{\partial z}+\frac{\partial^3 f^i}{\partial x^2\partial z}\frac{\partial g^3}{\partial z}\right)\frac{\partial g^1}{\partial x}\frac{\partial g^1}{\partial y}\\&\quad+\frac{\partial^2 f^i}{\partial x^2}\frac{\partial^2 g^1}{\partial x\partial z}\frac{\partial g^1}{\partial y}+\frac{\partial^2 f^i}{\partial x^2}\frac{\partial g^1}{\partial x}\frac{\partial^2 g^1}{\partial y\partial z}\\&\quad+\left(\frac{\partial^3 f^i}{\partial x\partial y^2}\frac{\partial g^1}{\partial z}+\frac{\partial^3 f^i}{\partial y^3}\frac{\partial g^2}{\partial z}+\frac{\partial^3 f^i}{\partial y^2\partial z}\frac{\partial g^3}{\partial z}\right)\frac{\partial g^2}{\partial x}\frac{\partial g^2}{\partial y}\\&\quad+\frac{\partial^2 f^i}{\partial y^2}\frac{\partial^2 g^2}{\partial x\partial z}\frac{\partial g^2}{\partial y}+\frac{\partial^2 f^i}{\partial y^2}\frac{\partial g^2}{\partial x}\frac{\partial^2 g^2}{\partial y\partial z}\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\partial^3 f^i}{\partial x\partial z^2}\frac{\partial g^1}{\partial z}+\frac{\partial^3 f^i}{\partial y\partial z^2}\frac{\partial g^2}{\partial z}+\frac{\partial^3 f^i}{\partial z^3}\frac{\partial g^3}{\partial z}\right)\frac{\partial g^3}{\partial x}\frac{\partial g^3}{\partial y}\\
&+\frac{\partial^2 f^i}{\partial z^2}\frac{\partial^2 g^3}{\partial x\partial z}\frac{\partial g^3}{\partial y}+\frac{\partial^2 f^i}{\partial z^2}\frac{\partial g^3}{\partial x}\frac{\partial^2 g^3}{\partial y\partial z}\\
&+\left(\frac{\partial^3 f^i}{\partial x^2\partial y}\frac{\partial g^1}{\partial z}+\frac{\partial^3 f^i}{\partial x\partial y^2}\frac{\partial g^2}{\partial z}+\frac{\partial^3 f^i}{\partial x\partial y\partial z}\frac{\partial g^3}{\partial z}\right)\left(\frac{\partial g^1}{\partial x}\frac{\partial g^2}{\partial y}+\frac{\partial g^1}{\partial y}\frac{\partial g^2}{\partial x}\right)\\
&+\frac{\partial^2 f^i}{\partial x\partial y}\left(\frac{\partial^2 g^1}{\partial x\partial z}\frac{\partial g^2}{\partial y}+\frac{\partial g^1}{\partial x}\frac{\partial^2 g^2}{\partial y\partial z}+\frac{\partial^2 g^1}{\partial y\partial z}\frac{\partial g^2}{\partial x}+\frac{\partial g^1}{\partial y}\frac{\partial^2 g^2}{\partial x\partial z}\right)\\
&+\left(\frac{\partial^3 f^i}{\partial x^2\partial z}\frac{\partial g^1}{\partial z}+\frac{\partial^3 f^i}{\partial x\partial y\partial z}\frac{\partial g^2}{\partial z}+\frac{\partial^3 f^i}{\partial x\partial z^2}\frac{\partial g^3}{\partial z}\right)\left(\frac{\partial g^1}{\partial x}\frac{\partial g^3}{\partial y}+\frac{\partial g^1}{\partial y}\frac{\partial g^3}{\partial x}\right)\\
&+\frac{\partial^2 f^i}{\partial x\partial z}\left(\frac{\partial^2 g^1}{\partial x\partial z}\frac{\partial g^3}{\partial y}+\frac{\partial g^1}{\partial x}\frac{\partial^2 g^3}{\partial y\partial z}+\frac{\partial^2 g^1}{\partial y\partial z}\frac{\partial g^3}{\partial x}+\frac{\partial g^1}{\partial y}\frac{\partial^2 g^3}{\partial x\partial z}\right)\\
&+\left(\frac{\partial^3 f^i}{\partial x\partial y\partial z}\frac{\partial g^1}{\partial z}+\frac{\partial^3 f^i}{\partial y^2\partial z}\frac{\partial g^2}{\partial z}+\frac{\partial^3 f^i}{\partial y\partial z^2}\frac{\partial g^3}{\partial z}\right)\left(\frac{\partial g^2}{\partial x}\frac{\partial g^3}{\partial y}+\frac{\partial g^2}{\partial y}\frac{\partial g^3}{\partial x}\right)\\
&+\frac{\partial^2 f^i}{\partial y\partial z}\left(\frac{\partial^2 g^2}{\partial x\partial z}\frac{\partial g^3}{\partial y}+\frac{\partial g^2}{\partial x}\frac{\partial^2 g^3}{\partial y\partial z}+\frac{\partial^2 g^2}{\partial y\partial z}\frac{\partial g^3}{\partial x}+\frac{\partial g^2}{\partial y}\frac{\partial^2 g^3}{\partial x\partial z}\right)\\
&+\left(\frac{\partial^2 f^i}{\partial x^2}\frac{\partial g^1}{\partial z}+\frac{\partial^2 f^i}{\partial x\partial y}\frac{\partial g^2}{\partial z}+\frac{\partial^2 f^i}{\partial x\partial z}\frac{\partial g^3}{\partial z}\right)\frac{\partial^2 g^1}{\partial x\partial y}+\frac{\partial f^i}{\partial x}\frac{\partial^3 g^1}{\partial x\partial y\partial z}\\
&+\left(\frac{\partial^2 f^i}{\partial x\partial y}\frac{\partial g^1}{\partial z}+\frac{\partial^2 f^i}{\partial y^2}\frac{\partial g^2}{\partial z}+\frac{\partial^2 f^i}{\partial y\partial z}\frac{\partial g^3}{\partial z}\right)\frac{\partial^2 g^2}{\partial x\partial y}+\frac{\partial f^i}{\partial y}\frac{\partial^3 g^2}{\partial x\partial y\partial z}\\
&+\left(\frac{\partial^2 f^i}{\partial x\partial z}\frac{\partial g^1}{\partial z}+\frac{\partial^2 f^i}{\partial y\partial z}\frac{\partial g^2}{\partial z}+\frac{\partial^2 f^i}{\partial z^2}\frac{\partial g^3}{\partial z}\right)\frac{\partial^2 g^3}{\partial x\partial y}+\frac{\partial f^i}{\partial z}\frac{\partial^3 g^3}{\partial x\partial y\partial z}
\end{aligned}
$$

from where we get the final form

$$
\begin{aligned}
\frac{\partial^3 h^i}{\partial x\partial y\partial z}&=\frac{\partial^3 f^i}{\partial x^3}\frac{\partial g^1}{\partial x}\frac{\partial g^1}{\partial y}\frac{\partial g^1}{\partial z}+\frac{\partial^3 f^i}{\partial y^3}\frac{\partial g^2}{\partial x}\frac{\partial g^2}{\partial y}\frac{\partial g^2}{\partial z}+\frac{\partial^3 f^i}{\partial z^3}\frac{\partial g^3}{\partial x}\frac{\partial g^3}{\partial y}\frac{\partial g^3}{\partial z}\\
&+\frac{\partial^3 f^i}{\partial x^2\partial y}\left(\frac{\partial g^1}{\partial x}\frac{\partial g^1}{\partial y}\frac{\partial g^2}{\partial z}+\frac{\partial g^1}{\partial y}\frac{\partial g^1}{\partial z}\frac{\partial g^2}{\partial x}+\frac{\partial g^1}{\partial x}\frac{\partial g^1}{\partial z}\frac{\partial g^2}{\partial y}\right)\\
&+\frac{\partial^3 f^i}{\partial x^2\partial z}\left(\frac{\partial g^1}{\partial x}\frac{\partial g^1}{\partial y}\frac{\partial g^3}{\partial z}+\frac{\partial g^1}{\partial y}\frac{\partial g^1}{\partial z}\frac{\partial g^3}{\partial x}+\frac{\partial g^1}{\partial x}\frac{\partial g^1}{\partial z}\frac{\partial g^3}{\partial y}\right)\\
&+\frac{\partial^3 f^i}{\partial x\partial y^2}\left(\frac{\partial g^1}{\partial x}\frac{\partial g^2}{\partial y}\frac{\partial g^2}{\partial z}+\frac{\partial g^1}{\partial y}\frac{\partial g^2}{\partial z}\frac{\partial g^2}{\partial x}+\frac{\partial g^1}{\partial x}\frac{\partial g^2}{\partial z}\frac{\partial g^2}{\partial y}\right)\\
&+\frac{\partial^3 f^i}{\partial x\partial z^2}\left(\frac{\partial g^1}{\partial x}\frac{\partial g^3}{\partial y}\frac{\partial g^3}{\partial z}+\frac{\partial g^1}{\partial y}\frac{\partial g^3}{\partial z}\frac{\partial g^3}{\partial x}+\frac{\partial g^1}{\partial x}\frac{\partial g^3}{\partial z}\frac{\partial g^3}{\partial y}\right)\\
&+\frac{\partial^3 f^i}{\partial y^2\partial z}\left(\frac{\partial g^2}{\partial x}\frac{\partial g^2}{\partial y}\frac{\partial g^3}{\partial z}+\frac{\partial g^2}{\partial y}\frac{\partial g^2}{\partial z}\frac{\partial g^3}{\partial x}+\frac{\partial g^2}{\partial x}\frac{\partial g^2}{\partial z}\frac{\partial g^3}{\partial y}\right)
\end{aligned}
$$

$$+\frac{\partial^3 f^i}{\partial y \partial z^2}\left(\frac{\partial g^2}{\partial x}\frac{\partial g^3}{\partial y}\frac{\partial g^3}{\partial z}+\frac{\partial g^2}{\partial y}\frac{\partial g^3}{\partial z}\frac{\partial g^3}{\partial x}+\frac{\partial g^2}{\partial x}\frac{\partial g^3}{\partial z}\frac{\partial g^3}{\partial y}\right)$$
$$+\frac{\partial^3 f^i}{\partial x \partial y \partial z}\left(\frac{\partial g^1}{\partial x}\frac{\partial g^2}{\partial y}\frac{\partial g^3}{\partial z}+\frac{\partial g^1}{\partial y}\frac{\partial g^2}{\partial x}\frac{\partial g^3}{\partial z}+\frac{\partial g^1}{\partial x}\frac{\partial g^2}{\partial z}\frac{\partial g^3}{\partial y}\right.$$
$$\left.+\frac{\partial g^1}{\partial y}\frac{\partial g^2}{\partial z}\frac{\partial g^3}{\partial x}+\frac{\partial g^1}{\partial z}\frac{\partial g^2}{\partial x}\frac{\partial g^3}{\partial y}+\frac{\partial g^1}{\partial z}\frac{\partial g^2}{\partial y}\frac{\partial g^3}{\partial x}\right)$$
$$+\frac{\partial^2 f^i}{\partial x^2}\left(\frac{\partial g^1}{\partial z}\frac{\partial^2 g^1}{\partial x \partial y}+\frac{\partial g^1}{\partial y}\frac{\partial^2 g^1}{\partial x \partial z}+\frac{\partial g^1}{\partial x}\frac{\partial^2 g^1}{\partial y \partial z}\right)$$
$$+\frac{\partial^2 f^i}{\partial y^2}\left(\frac{\partial g^2}{\partial z}\frac{\partial^2 g^2}{\partial x \partial y}+\frac{\partial g^2}{\partial y}\frac{\partial^2 g^2}{\partial x \partial z}+\frac{\partial g^2}{\partial x}\frac{\partial^2 g^2}{\partial y \partial z}\right)$$
$$+\frac{\partial^2 f^i}{\partial z^2}\left(\frac{\partial g^3}{\partial z}\frac{\partial^2 g^3}{\partial x \partial y}+\frac{\partial g^3}{\partial y}\frac{\partial^2 g^3}{\partial x \partial z}+\frac{\partial g^3}{\partial x}\frac{\partial^2 g^3}{\partial y \partial z}\right)$$
$$+\frac{\partial^2 f^i}{\partial x \partial y}\left(\frac{\partial g^2}{\partial z}\frac{\partial^2 g^1}{\partial x \partial y}+\frac{\partial g^1}{\partial z}\frac{\partial^2 g^2}{\partial x \partial y}+\frac{\partial g^1}{\partial y}\frac{\partial^2 g^2}{\partial x \partial z}\right.$$
$$\left.+\frac{\partial g^2}{\partial y}\frac{\partial^2 g^1}{\partial x \partial z}+\frac{\partial g^2}{\partial x}\frac{\partial^2 g^1}{\partial y \partial z}+\frac{\partial g^1}{\partial x}\frac{\partial^2 g^2}{\partial y \partial z}\right)$$
$$+\frac{\partial^2 f^i}{\partial x \partial z}\left(\frac{\partial g^1}{\partial z}\frac{\partial^2 g^3}{\partial x \partial y}+\frac{\partial g^3}{\partial z}\frac{\partial^2 g^1}{\partial x \partial y}+\frac{\partial g^1}{\partial y}\frac{\partial^2 g^3}{\partial x \partial z}\right.$$
$$\left.+\frac{\partial g^3}{\partial y}\frac{\partial^2 g^1}{\partial x \partial z}+\frac{\partial g^3}{\partial x}\frac{\partial^2 g^1}{\partial y \partial z}+\frac{\partial g^1}{\partial x}\frac{\partial^2 g^3}{\partial y \partial z}\right)$$
$$+\frac{\partial^2 f^i}{\partial y \partial z}\left(\frac{\partial g^3}{\partial z}\frac{\partial^2 g^2}{\partial x \partial y}+\frac{\partial g^2}{\partial z}\frac{\partial^2 g^3}{\partial x \partial y}+\frac{\partial g^2}{\partial y}\frac{\partial^2 g^3}{\partial x \partial z}\right.$$
$$\left.+\frac{\partial g^3}{\partial y}\frac{\partial^2 g^2}{\partial x \partial z}+\frac{\partial g^3}{\partial x}\frac{\partial^2 g^2}{\partial y \partial z}+\frac{\partial g^2}{\partial x}\frac{\partial^2 g^3}{\partial y \partial z}\right)$$
$$+\frac{\partial f^i}{\partial x}\frac{\partial^3 g^1}{\partial x \partial y \partial z}+\frac{\partial f^i}{\partial y}\frac{\partial^3 g^2}{\partial x \partial y \partial z}+\frac{\partial f^i}{\partial z}\frac{\partial^3 g^3}{\partial x \partial y \partial z}.$$

If we evaluate this partial derivative at the origin, we obtain the coordinates of the composed jet $j^3h = j^3f \circ j^3g$. Denote $a^i = f^i(\mathbf{0})$, $b^i = g^i(\mathbf{0})$ and $c^i = h^i(\mathbf{0})$. By using the notation from the previous section, i.e.,

$$\frac{\partial^3 h^i}{\partial x \partial y \partial z}(\mathbf{0}) = c^i_{123},$$

we obtain the desired result

$$\begin{aligned} c^i_{123} &= a^i_{111}b^1_1b^1_2b^1_3 + a^i_{222}b^2_1b^2_2b^2_3 + a^i_{333}b^3_1b^3_2b^3_3 \\ &\quad + a^i_{112}(b^1_1b^1_2b^2_3 + b^1_2b^1_3b^2_1 + b^1_1b^1_3b^2_2) \\ &\quad + a^i_{113}(b^1_1b^1_2b^3_3 + b^1_2b^1_3b^3_1 + b^1_1b^1_3b^3_2) \\ &\quad + a^i_{122}(b^1_3b^2_1b^2_2 + b^1_2b^2_1b^2_3 + b^1_1b^2_2b^2_3) \end{aligned}$$

$$
\begin{aligned}
&+ a^i_{133}(b^1_3 b^3_1 b^3_2 + b^1_2 b^3_1 b^3_3 + b^1_1 b^3_2 b^3_3) \\
&+ a^i_{223}(b^2_1 b^2_2 b^3_3 + b^2_2 b^2_3 b^3_1 + b^2_1 b^2_3 b^3_2) \\
&+ a^i_{233}(b^2_3 b^3_1 b^3_2 + b^2_2 b^3_1 b^3_3 + b^2_1 b^3_2 b^3_3) \\
&+ a^i_{123}(b^1_2 b^2_1 b^3_3 + b^1_1 b^2_2 b^3_3 + b^1_2 b^2_3 b^3_1 + b^1_1 b^2_3 b^3_2 + b^1_3 b^2_2 b^3_1 + b^1_3 b^2_1 b^3_2) \\
&+ a^i_{11}(b^1_3 b^1_{12} + b^1_2 b^1_{13} + b^1_1 b^1_{23}) + a^i_{22}(b^2_3 b^2_{12} + b^2_2 b^2_{13} + b^2_1 b^2_{23}) \\
&+ a^i_{33}(b^3_3 b^3_{12} + b^3_2 b^3_{13} + b^3_1 b^3_{23}) \\
&+ a^i_{12}(b^2_3 b^1_{12} + b^1_3 b^2_{12} + b^1_2 b^2_{13} + b^2_1 b^1_{23} + b^2_2 b^1_{13} + b^1_1 b^2_{23}) \\
&+ a^i_{13}(b^1_3 b^3_{12} + b^3_3 b^1_{12} + b^1_2 b^3_{13} + b^3_1 b^1_{23} + b^3_2 b^1_{13} + b^1_1 b^3_{23}) \\
&+ a^i_{23}(b^3_3 b^2_{12} + b^2_3 b^3_{12} + b^2_2 b^3_{13} + b^3_1 b^2_{23} + b^3_2 b^2_{13} + b^2_1 b^3_{23}) \\
&+ a^i_1 b^1_{123} + a^i_2 b^2_{123} + a^i_3 b^3_{123}.
\end{aligned}
$$

The fourth column of matrices C, E and F can now be derived in the following way. The first row of the representation matrix is given, respectively, by the elements $a^1_1, a^1_2, a^1_3, a^1_{11}, a^1_{22}, a^1_{33}, a^1_{12}, a^1_{13}, a^1_{23}, a^1_{111}, a^1_{222}, a^1_{333}, a^1_{123}, a^1_{112}, a^1_{113}, a^1_{122}, a^1_{133}, a^1_{223}, a^1_{233}$. Denote this matrix by α. Element $c^1_{123} \in \gamma$ of the matrix multiplication $\gamma = \alpha \cdot \beta$ can be obtained by multiplying the first row of the representation matrix α with the thirteenth column of another representation matrix β. By checking the result for the derived element c^1_{123}, it is easy to see that such a column has the form

$$
\beta_{13} = \begin{pmatrix}
b^1_{123} \\
b^2_{123} \\
b^3_{123} \\
b^1_3 b^1_{12} + b^1_2 b^1_{13} + b^1_1 b^1_{23} \\
b^2_3 b^2_{12} + b^2_2 b^2_{13} + b^2_1 b^2_{23} \\
b^3_3 b^3_{12} + b^3_2 b^3_{13} + b^3_1 b^3_{23} \\
b^2_3 b^1_{12} + b^1_3 b^2_{12} + b^1_2 b^2_{13} + b^2_1 b^1_{23} + b^2_2 b^1_{13} + b^1_1 b^2_{23} \\
b^1_3 b^3_{12} + b^3_3 b^1_{12} + b^1_2 b^3_{13} + b^3_1 b^1_{23} + b^3_2 b^1_{13} + b^1_1 b^3_{23} \\
b^3_3 b^2_{12} + b^2_3 b^3_{12} + b^2_2 b^3_{13} + b^3_1 b^2_{23} + b^3_2 b^2_{13} + b^2_1 b^3_{23} \\
b^1_1 b^1_2 b^1_3 \\
b^2_1 b^2_2 b^2_3 \\
b^3_1 b^3_2 b^3_3 \\
b^1_2 b^2_1 b^3_3 + b^1_1 b^2_2 b^3_3 + b^1_2 b^2_3 b^3_1 + b^1_1 b^2_3 b^3_2 + b^1_3 b^2_2 b^3_1 + b^1_3 b^2_1 b^3_2 \\
b^1_1 b^1_2 b^2_3 + b^1_2 b^1_3 b^2_1 + b^1_1 b^1_3 b^2_2 \\
b^1_1 b^1_2 b^3_3 + b^1_2 b^1_3 b^3_1 + b^1_1 b^1_3 b^3_2 \\
b^1_3 b^2_1 b^2_2 + b^1_2 b^2_1 b^2_3 + b^1_1 b^2_2 b^2_3 \\
b^1_3 b^3_1 b^3_2 + b^1_2 b^3_1 b^3_3 + b^1_1 b^3_2 b^3_3 \\
b^2_1 b^2_2 b^3_3 + b^2_2 b^2_3 b^3_1 + b^2_1 b^2_3 b^3_2 \\
b^2_3 b^3_1 b^3_2 + b^2_2 b^3_1 b^3_3 + b^2_1 b^3_2 b^3_3
\end{pmatrix}
$$

whose elements can be found in the fourth columns of matrices C, E and F, respectively. From this point it is easy to derive the rest of the columns. For example, to derive the eighteenth column of the representation matrix α (i.e., the ninth column of matrices C, E and F), it suffices to rewrite the lower indices 1 to 2 for all elements. Thus, the 18th column of the representation matrix α has the form

$$\alpha_{18} = \begin{pmatrix} a^1_{223} \\ a^2_{223} \\ a^3_{223} \\ a^1_{22}a^1_3 + 2a^1_2a^1_{23} \\ a^2_{22}a^2_3 + 2a^2_2a^2_{23} \\ a^3_{22}a^3_3 + 2a^3_2a^3_{23} \\ a^1_{22}a^2_3 + a^1_3a^2_{22} + 2a^1_{23}a^2_2 + 2a^1_2a^2_{23} \\ a^1_{22}a^3_3 + a^1_3a^3_{22} + 2a^1_{23}a^3_2 + 2a^1_2a^3_{23} \\ a^2_{22}a^3_3 + a^2_3a^3_{22} + 2a^2_{23}a^3_2 + 2a^2_2a^3_{23} \\ \left(a^1_2\right)^2 a^1_3 \\ \left(a^2_2\right)^2 a^2_3 \\ \left(a^3_2\right)^2 a^3_3 \\ 2a^1_2a^2_2a^3_3 + 2a^1_2a^2_3a^3_2 + 2a^1_3a^2_2a^3_2 \\ \left(a^1_2\right)^2 a^2_3 + 2a^1_2a^1_3a^2_2 \\ \left(a^1_2\right)^2 a^3_3 + 2a^1_2a^1_3a^3_2 \\ a^1_3\left(a^2_2\right)^2 + 2a^1_2a^2_2a^2_3 \\ a^1_3\left(a^3_2\right)^2 + 2a^1_2a^3_2a^3_3 \\ \left(a^2_2\right)^2 a^3_3 + 2a^2_2a^2_3a^3_2 \\ a^2_3\left(a^3_2\right)^2 + 2a^2_2a^3_2a^3_3 \end{pmatrix}.$$

□

3. The representation matrix of the jet groups G^3_2 and G^3_1. As many interesting results and PDEs are in two-dimensional space (ODEs in the case of one-dimensional space), we should also mention what the representation matrix looks like in a jet groups G^3_2 and G^3_1 as a corollary of the previous theorem.

Corollary 1. *The faithful matrix representation of the jet group G^3_2 has the form*

$$\alpha = \begin{pmatrix} A & B & C \end{pmatrix} \in M_9(\mathbb{R}),$$

where

$$A = \begin{pmatrix} a_1^1 & a_2^1 & a_{11}^1 & a_{22}^1 & a_{12}^1 \\ a_1^2 & a_2^2 & a_{11}^2 & a_{22}^2 & a_{12}^2 \\ 0 & 0 & \left(a_1^1\right)^2 & \left(a_1^1\right)^2 & a_1^1 a_2^1 \\ 0 & 0 & \left(a_1^2\right)^2 & \left(a_1^2\right)^2 & a_1^2 a_2^2 \\ 0 & 0 & 2a_1^1 a_1^2 & 2a_2^1 a_2^2 & a_1^1 a_2^2 + a_2^1 a_1^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} a_{111}^1 & a_{222}^1 \\ a_{111}^2 & a_{222}^2 \\ 3a_1^1 a_{11}^1 & 3a_2^1 a_{22}^1 \\ 3a_1^2 a_{11}^2 & 3a_2^2 a_{22}^2 \\ 3(a_{11}^1 a_1^2 + a_1^1 a_{11}^2) & 3(a_{22}^1 a_2^2 + a_2^1 a_{22}^2) \\ \left(a_1^1\right)^3 & \left(a_2^1\right)^3 \\ \left(a_1^2\right)^3 & \left(a_2^2\right)^3 \\ 3\left(a_1^1\right)^2 a_1^2 & 3\left(a_2^1\right)^2 a_2^2 \\ 3a_1^1\left(a_1^2\right)^2 & 3a_2^1\left(a_2^2\right)^2 \end{pmatrix},$$

$$C = \begin{pmatrix} a_{112}^1 & a_{122}^1 \\ a_{112}^2 & a_{122}^2 \\ a_{11}^1 a_2^1 + 2a_1^1 a_{12}^1 & a_{22}^1 a_1^1 + 2a_2^1 a_{12}^1 \\ a_{11}^2 a_2^2 + 2a_1^2 a_{12}^2 & a_{22}^2 a_1^2 + 2a_2^2 a_{12}^2 \\ a_{11}^1 a_2^2 + a_2^1 a_{11}^2 + 2a_{12}^1 a_1^2 + 2a_1^1 a_{12}^2 & a_{22}^1 a_1^2 + a_1^1 a_{22}^2 + 2a_{12}^1 a_2^2 + 2a_2^1 a_{12}^2 \\ \left(a_1^1\right)^2 a_2^1 & a_1^1\left(a_2^1\right)^2 \\ \left(a_1^2\right)^2 a_2^2 & a_1^2\left(a_2^2\right)^2 \\ \left(a_1^1\right)^2 a_2^2 + 2a_1^1 a_2^1 a_1^2 & \left(a_2^1\right)^2 a_1^2 + 2a_2^1 a_1^1 a_2^2 \\ a_2^1\left(a_1^2\right)^2 + 2a_1^1 a_1^2 a_2^2 & a_1^1\left(a_2^2\right)^2 + 2a_2^1 a_2^2 a_1^2 \end{pmatrix}.$$

Proof. To find the faithful matrix representation of a jet group G_2^3, it is enough to exclude all columns (rows) containing the index of the third variable from the G_3^3 representation matrix. □

Corollary 2. *The faithful matrix representation of the jet group G_1^3 has the form*

$$\alpha = \begin{pmatrix} a_1^1 & a_{11}^1 & a_{111}^1 \\ 0 & \left(a_1^1\right)^2 & 3a_1^1 a_{11}^1 \\ 0 & 0 & \left(a_1^1\right)^3 \end{pmatrix}.$$

Proof. Similarly to the proof of the previous corollary, it is enough to exclude all columns (rows) containing the index of the second variable from the G_2^3 representation matrix. □

4. The representation matrix of G_n^3. It is not difficult to generalize the matrix representation of G_3^3 to the case of general G_n^3 for $n \in \mathbb{N}$, $n \geq 4$. If we think of all rth partial derivatives, we can see that for a map of n variables, there are $\binom{n+r-1}{r}$ possibilities (number of combinations with repetition). The G_1^3 representation matrix therefore has to have $\binom{1}{1} + \binom{2}{2} + \binom{3}{3} = 3$ rows and columns, G_2^3 has to have $\binom{2}{1} + \binom{3}{2} + \binom{4}{3} = 9$ rows and columns, and finally, as we could see G_3^3 has to have $\binom{3}{1} + \binom{4}{2} + \binom{5}{3} = 19$ rows and columns. In the case of general $n \in \mathbb{N}$, G_n^3 must have

$$\binom{n}{1} + \binom{n+1}{2} + \binom{n+2}{3} = \frac{n}{6}(n^2 + 6n + 11) = \frac{n^3}{6} + n^2 + \frac{11n}{6}$$

rows and columns.

Corollary 3. *The faithful matrix representation of the jet group G_n^3, $n \in \mathbb{N}$ has the form*

$$\alpha = \begin{pmatrix} A & B & C \\ O & D & E \\ O & O & F \end{pmatrix} \in M_k(\mathbb{R}), \quad k = \frac{n^3}{6} + n^2 + \frac{11n}{6}$$

where O are zero matrices and for $m = n-1$

$$A = \begin{pmatrix} a_1^1 & a_2^1 & \dots & a_n^1 \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^n & a_2^n & \dots & a_n^n \end{pmatrix},$$

$$B = \begin{pmatrix} a_{11}^1 & a_{22}^1 & \dots & a_{nn}^1 & a_{12}^1 & a_{13}^1 & \dots & a_{pq}^1 & \dots & a_{mn}^1 \\ a_{11}^2 & a_{22}^2 & \dots & a_{nn}^2 & a_{12}^2 & a_{13}^2 & \dots & a_{pq}^2 & \dots & a_{mn}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{11}^n & a_{22}^n & \dots & a_{nn}^n & a_{12}^n & a_{13}^n & \dots & a_{pq}^n & \dots & a_{mn}^n \end{pmatrix},$$

$$\forall p, q \in \mathbb{N};\ 1 \leq p < q \leq n$$

$$C = \begin{pmatrix} a^1_{111} & a^1_{222} & \dots & a^1_{nnn} & a^1_{123} & \dots & a^1_{pqr} & a^1_{112} & \dots & a^1_{stu} & \dots & a^1_{mnn} \\ a^2_{111} & a^2_{222} & \dots & a^2_{nnn} & a^2_{123} & \dots & a^2_{pqr} & a^2_{112} & \dots & a^2_{stu} & \dots & a^2_{mnn} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a^n_{111} & a^n_{222} & \dots & a^n_{nnn} & a^n_{123} & \dots & a^n_{pqr} & a^n_{112} & \dots & a^n_{stu} & \dots & a^n_{mnn} \end{pmatrix},$$

$$\forall p,q,r \in \mathbb{N};\ 1 \le p < q < r \le n$$

$$\forall s,t,u \in \mathbb{N};\ (1 \le s = t < u \le n) \vee (1 \le s < t = u \le n)$$

$$D =$$

$$\begin{pmatrix} \left(a^1_1\right)^2 & \left(a^1_2\right)^2 & \dots & \left(a^1_n\right)^2 & a^1_1 a^1_2 & \dots & a^1_p a^1_q & \dots & a^1_m a^1_n \\ \left(a^2_1\right)^2 & \left(a^2_2\right)^2 & \dots & \left(a^2_n\right)^2 & a^2_1 a^2_2 & \dots & a^2_p a^2_q & \dots & a^2_m a^2_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (a^n_1)^2 & (a^n_2)^2 & \dots & (a^n_n)^2 & a^n_1 a^n_2 & \dots & a^n_p a^n_q & \dots & a^n_m a^n_n \\ 2a^1_1 a^2_1 & 2a^1_2 a^2_2 & \dots & 2a^1_n a^2_n & a^1_1 a^2_2 + a^1_2 a^2_1 & \dots & a^1_p a^2_q + a^1_q a^2_p & \dots & a^1_m a^2_n + a^1_n a^2_m \\ 2a^1_1 a^3_1 & 2a^1_2 a^3_2 & \dots & 2a^1_n a^3_n & a^1_1 a^3_2 + a^1_2 a^3_1 & \dots & a^1_p a^3_q + a^1_q a^3_p & \dots & a^1_m a^3_n + a^1_n a^3_m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 2a^p_1 a^q_1 & 2a^p_2 a^q_2 & \dots & 2a^p_n a^q_n & a^p_1 a^q_2 + a^p_2 a^q_1 & \dots & a^p_p a^q_q + a^p_q a^q_p & \dots & a^p_m a^q_n + a^p_n a^q_m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 2a^m_1 a^n_1 & 2a^m_2 a^n_2 & \dots & 2a^m_n a^n_n & a^m_1 a^n_2 + a^m_2 a^n_1 & \dots & a^m_p a^n_q + a^m_q a^n_p & \dots & a^m_m a^n_n + a^m_n a^n_m \end{pmatrix}$$

$$\forall p,q \in \mathbb{N};\ 1 \le p < q \le n$$

$$E = \begin{pmatrix} a^1_i a^1_{jk} + a^1_j a^1_{ik} + a^1_k a^1_{ij} \\ a^2_i a^2_{jk} + a^2_j a^2_{ik} + a^2_k a^2_{ij} \\ \vdots \\ a^n_i a^n_{jk} + a^n_j a^n_{ik} + a^n_k a^n_{ij} \\ a^1_i a^2_{jk} + a^1_j a^2_{ik} + a^1_k a^2_{ij} + a^2_i a^1_{jk} + a^2_j a^1_{ik} + a^2_k a^1_{ij} \\ a^1_i a^3_{jk} + a^1_j a^3_{ik} + a^1_k a^3_{ij} + a^3_i a^1_{jk} + a^3_j a^1_{ik} + a^3_k a^1_{ij} \\ \vdots \\ a^p_i a^q_{jk} + a^p_j a^q_{ik} + a^p_k a^q_{ij} + a^q_i a^p_{jk} + a^q_j a^p_{ik} + a^q_k a^p_{ij} \\ \vdots \\ a^m_i a^n_{jk} + a^m_j a^n_{ik} + a^m_k a^n_{ij} + a^n_i a^m_{jk} + a^n_j a^m_{ik} + a^n_k a^m_{ij} \end{pmatrix},$$

$$\forall p,q \in \mathbb{N};\ 1 \le p < q \le n$$

$$F = \begin{pmatrix} a_i^1 a_j^1 a_k^1 \\ a_i^2 a_j^2 a_k^2 \\ \vdots \\ a_i^n a_j^n a_k^n \\ a_i^1 a_j^2 a_k^3 + a_i^1 a_k^2 a_j^3 + a_j^1 a_i^2 a_k^3 + a_j^1 a_k^2 a_i^3 + a_k^1 a_i^2 a_j^3 + a_k^1 a_j^2 a_i^3 \\ \vdots \\ a_i^p a_j^q a_k^r + a_i^p a_k^q a_j^r + a_j^p a_i^q a_k^r + a_j^p a_k^q a_i^r + a_k^p a_i^q a_j^r + a_k^p a_j^q a_i^r \\ a_i^1 a_j^1 a_k^2 + a_j^1 a_k^1 a_i^2 + a_i^1 a_k^1 a_j^2 \\ a_i^1 a_j^1 a_k^3 + a_j^1 a_k^1 a_i^3 + a_i^1 a_k^1 a_j^3 \\ \vdots \\ a_i^s a_j^t a_k^u + a_j^s a_k^t a_i^u + a_i^s a_k^t a_j^u \\ \vdots \\ a_i^m a_j^n a_k^n + a_j^m a_k^n a_i^n + a_i^m a_k^n a_j^n \end{pmatrix},$$

$$\forall p, q, r \in \mathbb{N};\ 1 \leq p < q < r \leq n$$

$$\forall s, t, u \in \mathbb{N};\ (1 \leq s = t < u \leq n) \vee (1 \leq s < t = u \leq n)$$

where indices ijk in matrices E and F respect the upper indices in matrix F (or equivalently lower indices in matrix C).

Proof. Matrices A, B and D of the representation matrix α form the group G_n^2, whose elements were derived and can be found in [1]. To derive elements of matrices C, E and F, it is enough to derive the arbitrary column of these matrices with three different lower indices $\alpha, \beta, \gamma \in \mathbb{N}$, $1 \leq \alpha < \beta < \gamma \leq n$. Consider maps $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(\mathbf{0}) = \mathbf{0}$, $g : \mathbb{R}^n \to \mathbb{R}^n$, $g(\mathbf{0}) = \mathbf{0}$ and the composition map $h = f \circ g = (f^i(g^1(x^1, \dots, x^n), \dots, g^n(x^1, \dots, x^n)))$ for $i = 1, 2, \dots, n$. Now, we need to calculate the partial derivative at the origin

$$\frac{\partial^3 h^i}{\partial x^\alpha \partial x^\beta \partial x^\gamma}(\mathbf{0}) = c^i_{\alpha\beta\gamma}.$$

The result is similar to the one in the proof of Theorem 1. We obtain

$$\begin{aligned} c^i_{\alpha\beta\gamma} &= a^i_{111} b^1_\alpha b^1_\beta b^1_\gamma + \ldots + a^i_{nnn} b^n_\alpha b^n_\beta b^n_\gamma + a^i_{112}(b^1_\alpha b^1_\beta b^2_\gamma + b^1_\beta b^1_\gamma b^2_\alpha + b^1_\alpha b^1_\gamma b^2_\beta) \\ &+ a^i_{113}(b^1_\alpha b^1_\beta b^3_\gamma + b^1_\beta b^1_\gamma b^3_\alpha + b^1_\alpha b^1_\gamma b^3_\beta) + \ldots \\ &+ a^i_{stu}(b^s_\alpha b^t_\beta b^u_\gamma + b^s_\beta b^t_\gamma b^u_\alpha + b^s_\alpha b^t_\gamma b^u_\beta) + \ldots \\ &+ a^i_{123}(b^1_\beta b^2_\alpha b^3_\gamma + b^1_\alpha b^2_\beta b^3_\gamma + b^1_\beta b^2_\gamma b^3_\alpha + b^1_\alpha b^2_\gamma b^3_\beta + b^1_\gamma b^2_\beta b^3_\alpha + b^1_\gamma b^2_\alpha b^3_\beta) + \ldots \\ &+ a^i_{pqr}(b^p_\beta b^q_\alpha b^r_\gamma + b^p_\alpha b^q_\beta b^r_\gamma + b^p_\beta b^q_\gamma b^r_\alpha + b^p_\alpha b^q_\gamma b^r_\beta + b^p_\gamma b^q_\beta b^r_\alpha + b^p_\gamma b^q_\alpha b^r_\beta) + \ldots \\ &+ a^i_{11}(b^1_\gamma b^1_{\alpha\beta} + b^1_\beta b^1_{\alpha\gamma} + b^1_\alpha b^1_{\beta\gamma}) + \ldots + a^i_{nn}(b^n_\gamma b^n_{\alpha\beta} + b^n_\beta b^n_{\alpha\gamma} + b^n_\alpha b^n_{\beta\gamma}) \\ &+ a^i_{12}(b^2_\gamma b^1_{\alpha\beta} + b^1_\gamma b^2_{\alpha\beta} + b^1_\beta b^2_{\alpha\gamma} + b^2_\alpha b^1_{\beta\gamma} + b^2_\beta b^1_{\alpha\gamma} + b^1_\alpha b^2_{\beta\gamma}) \\ &+ a^i_{13}(b^1_\gamma b^3_{\alpha\beta} + b^3_\gamma b^1_{\alpha\beta} + b^1_\beta b^3_{\alpha\gamma} + b^3_\alpha b^1_{\beta\gamma} + b^3_\beta b^1_{\alpha\gamma} + b^1_\alpha b^3_{\beta\gamma}) + \ldots \end{aligned}$$

$$+ a^i_{mn}(b^n_\gamma b^m_{\alpha\beta} + b^m_\gamma b^n_{\alpha\beta} + b^m_\beta b^n_{\alpha\gamma} + b^n_\alpha b^m_{\beta\gamma} + b^n_\beta b^m_{\alpha\gamma} + b^m_\alpha b^n_{\beta\gamma})$$
$$+ a^i_1 b^1_{\alpha\beta\gamma} + \ldots + a^i_n b^n_{\alpha\beta\gamma},$$
$$m = n - 1$$
$$\forall p, q, r \in \mathbb{N};\ 1 \leq p < q < r \leq n$$
$$\forall s, t, u \in \mathbb{N};\ (1 \leq s = t < u \leq n) \vee (1 \leq s < t = u \leq n).$$

If we again think of the element $c^i_{\alpha\beta\gamma}$ as a result of matrix multiplication, we obtain the column of the G^3_n representation matrix with lower indices $\alpha\beta\gamma$. Since the indices $\alpha\beta\gamma$ were arbitrarily chosen, we can describe any column of the G^3_n representation matrix from blocks C, E and F. □

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