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Generalized commutative quaternion polynomials of the Fibonacci type

Abstract. Generalized commutative quaternions is a number system which generalizes elliptic, parabolic and hyperbolic quaternions, bicomplex numbers, complex hyperbolic numbers and hyperbolic complex numbers. In this paper we introduce and study generalized commutative quaternion polynomials of the Fibonacci type.

1. Introduction and preliminaries. In 1843 ([3]), Hamilton introduced the set $\mathbb{H}$ of quaternions q of the form

$$q = x_0 + x_1 i + x_2 j + x_3 k$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

The above rules immediately show that multiplication of quaternions is not commutative. For this reason some quaternions algebra problems are not easy. In [16], Segre modified quaternions in such a way that they admit a commutative property in multiplication and consequently introduced commutative quaternions. The set of commutative quaternions is a 4-dimensional structure, contains zero-divisor and isotropic elements. In the

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last years theory of commutative quaternions is becoming interesting and many applications can be found. Many physical laws in classical, relativistic and quantum mechanics can be written nicely using quaternions. Quaternions appear not only in various areas of pure mathematics and physics but also in mechanics, see for example [2, 14, 15].

Noncommutative quaternions and commutative quaternions were generalized and studied recently, see for details the paper of Jafari and Yayli [9] where generalized quaternions were introduced and also results of Szynal-Liana and Włoch in [19], where generalized commutative quaternions were introduced and studied in the special subfamily of quaternions of the Fibonacci type.

In this paper we will study generalized commutative quaternions, so first let us recall necessary definitions.

Let $\mathbb{H}^c_{\alpha\beta}$ be the set of generalized commutative quaternions $\mathbf{x}$ of the form

$$\mathbf{x} = x_0 + x_1e_1 + x_2e_2 + x_3e_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$, quaternionic units e_1, e_2, e_3 satisfy the equalities

$$e_1^2 = \alpha, \quad e_2^2 = \beta, \quad e_3^2 = \alpha\beta,$$
$$e_1e_2 = e_2e_1 = e_3, \quad e_2e_3 = e_3e_2 = \beta e_1 \quad \text{and} \quad e_3e_1 = e_1e_3 = \alpha e_2,$$

and $\alpha, \beta \in \mathbb{R}$.

The generalized commutative quaternions generalize elliptic quaternions ($\alpha < 0$, $\beta = 1$), parabolic quaternions ($\alpha = 0$, $\beta = 1$), hyperbolic quaternions ($\alpha > 0$, $\beta = 1$), bicomplex numbers ($\alpha = -1$, $\beta = -1$), complex hyperbolic numbers ($\alpha = -1$, $\beta = 1$) and hyperbolic complex numbers ($\alpha = 1$, $\beta = -1$).

Horadam in [4] introduced the concept of Fibonacci and Lucas quaternions and many mathematicians focused on this theory. For example Iyer [8] gave relations between Fibonacci and Lucas quaternions. Iakin in [6, 7] introduced the concept of a higher order quaternion and established some identities for these quaternions. In [5], Horadam mentioned also the possibility of introducing another Fibonacci type quaternions.

Taking into account a number of papers related to this topic, we can indicate that in [1] generalized commutative Jacobsthal quaternions and generalized commutative Jacobsthal–Lucas quaternions were studied. Interesting results related to Fibonacci type quaternions can be found in papers of Kızılateş. In particular, in [11] Kızılateş et al. introduced and studied bicomplex generalized Tribonacci quaternions and in [10] incomplete Fibonacci and Lucas quaternions. Furthermore, in [13] Kızılateş and Kone introduced and studied higher order Fibonacci quaternions. In that papers combinatorial and algebraic properties of Fibonacci type quaternions were presented and also with respect to determinant of special matrices.

Generalized Fibonacci quaternions were examined by Swamy in [17]. Actually theory of the so-called Fibonacci type hypercomplex numbers, including also quaternions, is widely explored, see for details the bibliography of [18]. Fibonacci type sequences are generalized by considering Fibonacci type polynomials. Sequences defined by the second-order linear homogeneous recurrence equation of the form

$$f_n(x) = g(x)f_{n-1}(x) + h(x)f_{n-2}(x), \tag{1}$$

for $n \geq 2$ with fixed $f_0(x), f_1(x)$ are named as Fibonacci type (or Fibonacci-like) polynomials and their elements as polynomials of the Fibonacci type (or Fibonacci-like), respectively. The family of Fibonacci type polynomials defined by the equation (1) we will denote by $\mathcal{F}_n(x)$ and sequences belonging to $\mathcal{F}_n(x)$ we will denote by $F(f_0(x), f_1(x); g(x), h(x))$. The family $\mathcal{F}_n(x)$ includes the well-known sequences of polynomials and recently there have been so many studies of them in the literature. We list only classical Fibonacci type sequences of polynomials:

$F(0, 1; x, 1)$ – Fibonacci polynomials sequence $\{F_n(x)\}$,
$F(2, x; x, 1)$ – Lucas polynomials sequence $\{L_n(x)\}$,
$F(0, 1; 2x, 1)$ – Pell polynomials sequence $\{P_n(x)\}$,
$F(2, 2x; 2x, 1)$ – Pell–Lucas polynomials sequence $\{Q_n(x)\}$,
$F(0, 1; 1, 2x)$ – Jacobsthal polynomials sequence $\{J_n(x)\}$,
$F(2, x; 1, 2x)$ – Jacobsthal–Lucas polynomials sequence $\{j_n(x)\}$.

If $x = 1$, then we obtain classical Fibonacci type sequences and also we give their indication in *On-line Encyclopedia of Integer Sequences* which is a popular online database of integer sequences, see [21]:

$F(0, 1; 1, 1)$ – Fibonacci sequence $\{F_n\}$ (`https://oeis.org/A000045`)
$F(2, 1; 1, 1)$ – Lucas sequence $\{L_n\}$ (`https://oeis.org/A000032`)
$F(0, 1; 2, 1)$ – Pell sequence $\{P_n\}$ (`https://oeis.org/A000129`)
$F(2, 2; 2, 1)$ – Pell-Lucas sequence $\{Q_n\}$ (`https://oeis.org/A002203`)
$F(0, 1; 1, 2)$ – Jacobsthal sequence $\{J_n\}$ (`https://oeis.org/A001045`)
$F(2, 1; 1, 2)$ – Jacobsthal-Lucas sequence $\{j_n\}$ (`https://oeis.org/A014551`).

The first few Fibonacci type polynomials are given in Table 1.

n	0	1	2	3	4	5
$F_n(x)$	0	1	x	x^2+1	x^3+2x	x^4+3x^2+1
$L_n(x)$	2	x	x^2+2	x^3+3x	x^4+4x^2+2	x^5+5x^3+5x
$P_n(x)$	0	1	$2x$	$4x^2+1$	$8x^3+4x$	$16x^3+12x^2+1$
$Q_n(x)$	2	$2x$	$4x^2+2$	$8x^3+6x$	$16x^4+16x^2+2$	$32x^5+40x^3+10x$
$J_n(x)$	0	1	1	$2x+1$	$4x+1$	$4x^2+6x+1$
$j_n(x)$	2	x	$4x+1$	$6x+1$	$8x^2+8x+1$	$20x^2+10x+1$

Table 1. A few first words of Fibonacci type polynomials.

Table 2 gives a few first elements of sequences of the Fibonacci type.

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
L_n	2	1	3	4	7	11	18	29	47	76	123
P_n	0	1	2	5	12	29	70	169	408	985	2378
Q_n	2	2	6	14	34	82	198	478	1154	2786	6726
J_n	0	1	1	3	5	11	21	43	85	171	341
j_n	2	1	5	7	17	31	65	127	257	511	1025

Table 2. The values of numbers of the Fibonacci type.

Based on the properties of sequences defined by the second-order linear recurrence relations we can give direct formulas for Fibonacci type polynomials.

Binet formulas for Fibonacci and Lucas polynomials have the form

$$F_n(x) = \frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)} \tag{2}$$

and

$$L_n(x) = \gamma^n(x) + \delta^n(x),$$

where

$$\gamma(x) = \frac{1}{2}\left(x + \sqrt{x^2+4}\right), \quad \delta(x) = \frac{1}{2}\left(x - \sqrt{x^2+4}\right). \tag{3}$$

Binet formulas for Pell and Pell–Lucas polynomials have the form

$$P_n(x) = \frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)},$$

and

$$Q_n(x) = \gamma^n(x) + \delta^n(x)$$

where

$$\gamma(x) = x + \sqrt{x^2+1}, \quad \delta(x) = x - \sqrt{x^2+1}.$$

Binet formulas for Jacobsthal and Jacobsthal–Lucas polynomials have the form

$$J_n(x) = \frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)},$$

and

$$j_n(x) = \gamma^n(x) + \delta^n(x),$$

where

$$\gamma(x) = \frac{1}{2}\left(1 + \sqrt{8x+1}\right), \quad \delta(x) = \frac{1}{2}\left(1 - \sqrt{8x+1}\right).$$

Let $n \geq 0$ be an integer. The nth generalized commutative Fibonacci quaternion $gc\mathcal{F}_n$, the nth generalized commutative Lucas quaternion $gc\mathcal{L}_n$, the nth generalized commutative Pell quaternion $gc\mathcal{P}_n$, the nth generalized commutative Pell–Lucas quaternion $gc\mathcal{Q}_n$, the nth generalized commutative Jacobsthal quaternion $gc\mathcal{J}_n$ and the nth generalized commutative Jacobsthal–Lucas quaternion $gc\mathcal{JL}_n$ are defined as

$$\begin{aligned}
gc\mathcal{F}_n &= F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3,\\
gc\mathcal{L}_n &= L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3,\\
gc\mathcal{P}_n &= P_n + P_{n+1}e_1 + P_{n+2}e_2 + P_{n+3}e_3,\\
gc\mathcal{Q}_n &= Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3,\\
gc\mathcal{J}_n &= J_n + J_{n+1}e_1 + J_{n+2}e_2 + J_{n+3}e_3,\\
gc\mathcal{JL}_n &= j_n + j_{n+1}e_1 + j_{n+2}e_2 + j_{n+3}e_3,
\end{aligned}$$

respectively.

These quaternions are special types of generalized commutative Horadam quaternions defined in [19]. Now we will define generalized commutative quaternion polynomials of the Fibonacci type.

Let $n \geq 0$ be an integer and x be a real variable. The nth Fibonacci generalized commutative quaternion polynomial $gc\mathcal{F}_n(x)$, the nth Lucas generalized commutative quaternion polynomial $gc\mathcal{L}_n(x)$, the nth Pell generalized commutative quaternion polynomial $gc\mathcal{P}_n(x)$, the nth Pell–Lucas generalized commutative quaternion polynomial $gc\mathcal{Q}_n(x)$, the nth Jacobsthal generalized commutative quaternion polynomial $gc\mathcal{J}_n(x)$ and the nth Jacobsthal–Lucas generalized commutative quaternion polynomial $gc\mathcal{JL}_n(x)$ are defined by

$$\begin{aligned}
(4)\qquad gc\mathcal{F}_n(x) &= F_n(x) + F_{n+1}(x)e_1 + F_{n+2}(x)e_2 + F_{n+3}(x)e_3,\\
gc\mathcal{L}_n(x) &= L_n(x) + L_{n+1}(x)e_1 + L_{n+2}(x)e_2 + L_{n+3}(x)e_3,\\
gc\mathcal{P}_n(x) &= P_n(x) + P_{n+1}(x)e_1 + P_{n+2}(x)e_2 + P_{n+3}(x)e_3,\\
gc\mathcal{Q}_n(x) &= Q_n(x) + Q_{n+1}(x)e_1 + Q_{n+2}(x)e_2 + Q_{n+3}(x)e_3,
\end{aligned}$$

$$gc\mathcal{J}_n(x) = J_n(x) + J_{n+1}(x)e_1 + J_{n+2}(x)e_2 + J_{n+3}(x)e_3,$$
$$gc\mathcal{JL}_n(x) = j_n(x) + j_{n+1}(x)e_1 + j_{n+2}(x)e_2 + j_{n+3}(x)e_3,$$

respectively.

If $x = 1$, then $gc\mathcal{F}_n(1) = gc\mathcal{F}_n$, $gc\mathcal{L}_n(1) = gc\mathcal{L}_n$, $gc\mathcal{P}_n(1) = gc\mathcal{P}_n$, $gc\mathcal{Q}_n(1) = gc\mathcal{Q}_n$, $gc\mathcal{J}_n(1) = gc\mathcal{J}_n$ and $gc\mathcal{JL}_n(1) = gc\mathcal{JL}_n$.

In the next section we will give some properties of generalized commutative quaternion polynomials of the Fibonacci type.

2. Main results.

Theorem 1 (Binet type formula for Fibonacci generalized commutative quaternion polynomials). *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned} gc\mathcal{F}_n(x) &= \frac{\gamma^n(x)}{\gamma(x)-\delta(x)}\left(1+\gamma(x)e_1+\gamma^2(x)e_2+\gamma^3(x)e_3\right) \\ &\quad - \frac{\delta^n(x)}{\gamma(x)-\delta(x)}\left(1+\delta(x)e_1+\delta^2(x)e_2+\delta^3(x)e_3\right), \end{aligned} \tag{5}$$

where $\gamma(x)$, $\delta(x)$ are given by (3).

Proof. Using (4) and (2), we have

$$\begin{aligned} gc\mathcal{F}_n(x) &= F_n(x) + F_{n+1}(x)e_1 + F_{n+2}(x)e_2 + F_{n+3}(x)e_3 \\ &= \frac{\gamma^n(x)-\delta^n(x)}{\gamma(x)-\delta(x)} + \frac{\gamma^{n+1}(x)-\delta^{n+1}(x)}{\gamma(x)-\delta(x)}e_1 \\ &\quad + \frac{\gamma^{n+2}(x)-\delta^{n+2}(x)}{\gamma(x)-\delta(x)}e_2 + \frac{\gamma^{n+3}(x)-\delta^{n+3}(x)}{\gamma(x)-\delta(x)}e_3 \end{aligned}$$

and after calculations the result follows. □

For simplicity of notation let

$$\begin{aligned} \widehat{\gamma}(x) &= 1+\gamma(x)e_1+\gamma^2(x)e_2+\gamma^3(x)e_3, \\ \widehat{\delta}(x) &= 1+\delta(x)e_1+\delta^2(x)e_2+\delta^3(x)e_3. \end{aligned} \tag{6}$$

Theorem 2 (General bilinear index-reduction formula for Fibonacci generalized commutative quaternion polynomials). *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a+b=c+d$. Then*

$$\begin{aligned} & gc\mathcal{F}_a(x)\cdot gc\mathcal{F}_b(x) - gc\mathcal{F}_c(x)\cdot gc\mathcal{F}_d(x) \\ &= \frac{\left[\gamma^c(x)\delta^d(x)-\gamma^a(x)\delta^b(x)+\delta^c(x)\gamma^d(x)-\delta^a(x)\gamma^b(x)\right]\widehat{\gamma}(x)\widehat{\delta}(x)}{\left(\gamma(x)-\delta(x)\right)^2}, \end{aligned}$$

where $\gamma(x)$, $\delta(x)$ are given by (3) *and $\widehat{\gamma}(x)$, $\widehat{\delta}(x)$ are given by* (6).

Proof. Using (5), we have

$$
\begin{aligned}
&gc\mathcal{F}_a(x)\cdot gc\mathcal{F}_b(x)-gc\mathcal{F}_c(x)\cdot gc\mathcal{F}_d(x)\\
&=\frac{-\gamma^a(x)\delta^b(x)\widehat{\gamma}(x)\widehat{\delta}(x)-\delta^a(x)\gamma^b(x)\widehat{\delta}(x)\widehat{\gamma}(x)}{(\gamma(x)-\delta(x))^2}\\
&\quad+\frac{\gamma^c(x)\delta^d(x)\widehat{\gamma}(x)\widehat{\delta}(x)+\delta^c(x)\gamma^d(x)\widehat{\delta}(x)\widehat{\gamma}(x)}{(\gamma(x)-\delta(x))^2}\\
&=\frac{\left[\gamma^c(x)\delta^d(x)-\gamma^a(x)\delta^b(x)+\delta^c(x)\gamma^d(x)-\delta^a(x)\gamma^b(x)\right]\widehat{\gamma}(x)\widehat{\delta}(x)}{(\gamma(x)-\delta(x))^2},
\end{aligned}
$$

which ends the proof. □

For special values of a, b, c, d we obtain Catalan, Cassini, Vajda and d'Ocagne type identity, respectively.

Corollary 3 (Catalan type identity for Fibonacci generalized commutative quaternion polynomials). *Let $n\geq 0$, $r\geq 0$ be integers such that $n\geq r$. Then*

$$
\begin{aligned}
&gc\mathcal{F}_{n+r}(x)\cdot gc\mathcal{F}_{n-r}(x)-(gc\mathcal{F}_n(x))^2\\
&=\frac{\gamma^n(x)\delta^n(x)\widehat{\gamma}(x)\widehat{\delta}(x)}{(\gamma(x)-\delta(x))^2}\left[2-\left(\frac{\gamma(x)}{\delta(x)}\right)^r-\left(\frac{\delta(x)}{\gamma(x)}\right)^r\right],
\end{aligned}
$$

where $\gamma(x)$, $\delta(x)$ are given by (3) *and $\widehat{\gamma}(x)$, $\widehat{\delta}(x)$ are given by* (6).

Corollary 4 (Cassini type identity for Fibonacci generalized commutative quaternion polynomials). *Let $n\geq 1$ be an integer. Then*

$$
\begin{aligned}
&gc\mathcal{F}_{n+1}(x)\cdot gc\mathcal{F}_{n-1}(x)-(gc\mathcal{F}_n(x))^2\\
&=\frac{\gamma^n(x)\delta^n(x)\widehat{\gamma}(x)\widehat{\delta}(x)}{(\gamma(x)-\delta(x))^2}\left[2-\frac{\gamma(x)}{\delta(x)}-\frac{\delta(x)}{\gamma(x)}\right],
\end{aligned}
$$

where $\gamma(x)$, $\delta(x)$ are given by (3) *and $\widehat{\gamma}(x)$, $\widehat{\delta}(x)$ are given by* (6).

Corollary 5 (Vajda type identity for Fibonacci generalized commutative quaternion polynomials). *Let $n\geq 0$, $m\geq 0$, $p\geq 0$ be integers such that $n\geq p$. Then*

$$
\begin{aligned}
&gc\mathcal{F}_{m+p}(x)\cdot gc\mathcal{F}_{n-p}(x)-gc\mathcal{F}_m(x)\cdot gc\mathcal{F}_n(x)\\
&=\frac{\left[\gamma^m(x)\delta^n(x)\left(1-\left(\frac{\gamma(x)}{\delta(x)}\right)^p\right)+\delta^m(x)\gamma^n(x)\left(1-\left(\frac{\delta(x)}{\gamma(x)}\right)^p\right)\right]\widehat{\gamma}(x)\widehat{\delta}(x)}{(\gamma(x)-\delta(x))^2},
\end{aligned}
$$

where $\gamma(x)$, $\delta(x)$ are given by (3) *and $\widehat{\gamma}(x)$, $\widehat{\delta}(x)$ are given by* (6).

Corollary 6 (d'Ocagne type identity for Fibonacci generalized commutative quaternion polynomials). *Let $n \geq 0$, $m \geq 0$ be integers such that $n \geq m$. Then*

$$\begin{aligned} &gc\mathcal{F}_n(x) \cdot gc\mathcal{F}_{m+1}(x) - gc\mathcal{F}_{n+1}(x) \cdot gc\mathcal{F}_m(x) \\ &= \frac{[\gamma^n(x)\delta^m(x) - \delta^n(x)\gamma^m(x)]\,\widehat{\gamma}(x)\widehat{\delta}(x)}{\gamma(x) - \delta(x)}, \end{aligned}$$

where $\gamma(x)$, $\delta(x)$ are given by (3) *and $\widehat{\gamma}(x)$, $\widehat{\delta}(x)$ are given by* (6)*.*

In the same way we can prove results for Lucas generalized commutative quaternion polynomials.

Theorem 7 (Binet type formula for Lucas generalized commutative quaternion polynomials). *Let $n \geq 0$ be an integer. Then*

$$gc\mathcal{L}_n(x) = \gamma^n(x)\widehat{\gamma}(x) + \delta^n(x)\widehat{\delta}(x),$$

where $\gamma(x)$, $\delta(x)$ are given by (3) *and $\widehat{\gamma}(x)$, $\widehat{\delta}(x)$ are given by* (6)*.*

Theorem 8 (General bilinear index-reduction formula for Lucas generalized commutative quaternion polynomials). *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Then*

$$\begin{aligned} &gc\mathcal{L}_a(x) \cdot gc\mathcal{L}_b(x) - gc\mathcal{L}_c(x) \cdot gc\mathcal{L}_d(x) \\ &= \left[\gamma^a(x)\delta^b(x) + \delta^a(x)\gamma^b(x) - \gamma^c(x)\delta^d(x) - \delta^c(x)\gamma^d(x)\right]\widehat{\gamma}(x)\widehat{\delta}(x), \end{aligned}$$

where $\gamma(x)$, $\delta(x)$ are given by (3) *and $\widehat{\gamma}(x)$, $\widehat{\delta}(x)$ are given by* (6)*.*

Corollary 9 (Catalan type identity for Lucas generalized commutative quaternion polynomials). *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$\begin{aligned} &gc\mathcal{L}_{n+r}(x) \cdot gc\mathcal{L}_{n-r}(x) - (gc\mathcal{L}_n(x))^2 \\ &= \gamma^n(x)\delta^n(x)\left[\left(\frac{\gamma(x)}{\delta(x)}\right)^r + \left(\frac{\delta(x)}{\gamma(x)}\right)^r - 2\right]\widehat{\gamma}(x)\widehat{\delta}(x), \end{aligned}$$

where $\gamma(x)$, $\delta(x)$ are given by (3) *and $\widehat{\gamma}(x)$, $\widehat{\delta}(x)$ are given by* (6)*.*

Corollary 10 (Cassini type identity for Lucas generalized commutative quaternion polynomials). *Let $n \geq 1$ be an integer. Then*

$$\begin{aligned} &gc\mathcal{L}_{n+1}(x) \cdot gc\mathcal{L}_{n-1}(x) - (gc\mathcal{L}_n(x))^2 \\ &= \gamma^n(x)\delta^n(x)\left[\frac{\gamma(x)}{\delta(x)} + \frac{\delta(x)}{\gamma(x)} - 2\right]\widehat{\gamma}(x)\widehat{\delta}(x), \end{aligned}$$

where $\gamma(x)$, $\delta(x)$ are given by (3) *and $\widehat{\gamma}(x)$, $\widehat{\delta}(x)$ are given by* (6)*.*

Corollary 11 (Vajda type identity for Lucas generalized commutative quaternion polynomials). *Let $n \geq 0$, $m \geq 0$, $p \geq 0$ be integers such that $n \geq p$. Then*

$$gc\mathcal{L}_{m+p}(x) \cdot gc\mathcal{L}_{n-p}(x) - gc\mathcal{L}_m(x) \cdot gc\mathcal{L}_n(x)$$
$$= \left[\gamma^m(x)\delta^n(x)\left(\left(\frac{\gamma(x)}{\delta(x)}\right)^p - 1\right) + \delta^m(x)\gamma^n(x)\left(\left(\frac{\delta(x)}{\gamma(x)}\right)^p - 1\right)\right]\widehat{\gamma}(x)\widehat{\delta}(x),$$

where $\gamma(x)$, $\delta(x)$ are given by (3) *and $\widehat{\gamma}(x)$, $\widehat{\delta}(x)$ are given by* (6).

Corollary 12 (d'Ocagne type identity for Lucas generalized commutative quaternion polynomials). *Let $n \geq 0$, $m \geq 0$ be integers such that $n \geq m$. Then*

$$\begin{aligned} &gc\mathcal{L}_n(x) \cdot gc\mathcal{L}_{m+1}(x) - gc\mathcal{L}_{n+1}(x) \cdot gc\mathcal{L}_m(x) \\ &= \left(\gamma^n(x)\delta^m(x) - \delta^n(x)\gamma^m(x)\right)\left(\delta(x) - \gamma(x)\right)\widehat{\gamma}(x)\widehat{\delta}(x), \end{aligned}$$

where $\gamma(x)$, $\delta(x)$ are given by (3) *and $\widehat{\gamma}(x)$, $\widehat{\delta}(x)$ are given by* (6).

Based on the above methods we can give results for other classical Fibonacci type generalized commutative quaternion polynomials i.e. Pell, Pell–Lucas, Jacobsthal, Jacobsthal–Lucas generalized commutative quaternion polynomials.

Using (6), we have

$$\begin{aligned} \widehat{\gamma(x)}\widehat{\delta(x)} &= \widehat{\delta(x)}\widehat{\gamma(x)} \\ &= 1 + \gamma(x)\delta(x)\alpha + \gamma^2(x)\delta^2(x)\beta + \gamma^3(x)\delta^3(x)\alpha\beta \\ &\quad + \left(\gamma(x) + \delta(x) + \gamma^2(x)\delta^3(x)\beta + \gamma^3(x)\delta^2(x)\beta\right)e_1 \\ &\quad + \left(\gamma^2(x) + \delta^2(x) + \gamma(x)\delta^3(x)\alpha + \gamma^3(x)\delta(x)\alpha\right)e_2 \\ &\quad + \left(\gamma^3(x) + \delta^3(x) + \gamma(x)\delta^2(x) + \gamma^2(x)\delta(x)\right)e_3 \\ &= 1 + \gamma(x)\delta(x)\alpha + \left(\gamma(x)\delta(x)\right)^2\beta + \left(\gamma(x)\delta(x)\right)^3\alpha\beta \\ &\quad + \left(\gamma(x) + \delta(x)\right)\left(1 + \left(\gamma(x)\delta(x)\right)^2\beta\right)e_1 \\ &\quad + \left(\gamma^2(x) + \delta^2(x)\right)\left(1 + \gamma(x)\delta(x)\alpha\right)e_2 \\ &\quad + \left(\gamma^3(x) + \delta^3(x) + \gamma(x)\delta(x)\left(\delta(x) + \gamma(x)\right)\right)e_3. \end{aligned}$$

By simple calculations we obtain

$$\begin{aligned} \gamma^2(x) + \delta^2(x) &= (\gamma(x) + \delta(x))^2 - 2\gamma(x)\delta(x), \\ \gamma^3(x) + \delta^3(x) &= (\gamma(x) + \delta(x))^3 - 3\gamma(x)\delta(x)(\gamma(x) + \delta(x)). \end{aligned}$$

Dependencies between functions $\delta(x)$ and $\gamma(x)$ which appears in the above equations can be determined for special pairs of Fibonacci type polynomials sequences and we show that they are fixed polynomials for Fibonacci and

Lucas, Pell and Pell–Lucas and Jacobsthal and Jacobsthal–Lucas polynomials sequences. These polynomials are indicated in the following table.

	$\gamma(x)+\delta(x)$	$\gamma(x)\delta(x)$	$\gamma^2(x)+\delta^2(x)$	$\gamma^3(x)+\delta^3(x)$
$F_n(x),\ L_n(x)$	x	-1	x^2+2	x^3+3x
$P_n(x),\ Q_n(x)$	$2x$	-1	$4x^2+2$	$8x^3+6x$
$J_n(x),\ j_n(x)$	1	$-2x$	$1+4x$	$1+6x$

Table 3. Dependencies between $\delta(x)$ and $\gamma(x)$.

Concluding Remarks. In this paper, we introduced and discussed the concept of generalized commutative quaternion polynomials of the Fibonacci type. Presented results concern different families of quaternions. The problems analyzed in this paper can be still extended for other generalizations of quaternions and we believe that the obtained results may be a significant step in exploration of this field. The presented concept and results can have the potential to motivate further research on the subject of the Fibonacci finite operators quaternions and other Fibonacci hypercomplex numbers using methods and interesting results given for example in the papers [12, 20, 22].

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