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Counting holomorphic connections with a prescribed Ricci tensor

Abstract. How many holomorphic connections are there with a prescribed Ricci tensor? How many torsion-free holomorphic connections are there with a prescribed Ricci tensor? These questions are answered by using the holomorphic version of the Cauchy–Kowalevski theorem.

1. Introduction. In the present paper, given a holomorphic tensor field r of type $(0, 2)$ on a complex manifold (M, J), we use the holomorphic version of the Cauchy–Kowalevski theorem to describe all local holomorphic solutions ∇ of the equation

$$\mathcal{R}ic^{\nabla} = r \tag{1}$$

with unknown holomorphic connection ∇ on M, where $\mathcal{R}ic^{\nabla}$ is the Ricci tensor of ∇ (defined in Section 3). We also describe all local solutions of the equation (1) with unknown torsion-free holomorphic connection ∇.

Similar problems have been studied in many papers, e.g. [1, 2, 3, 5, 6, 8, 9]. For example, in [6], the author studied the existence of local solutions ∇ of the equation $\mathcal{R}ic^{\nabla} = r$ with unknown real analytical connection ∇ on a real analytical manifold M, where r is a symmetric real analytical tensor field of type $(0, 2)$ on M. In [3], using the Cauchy–Kowalevski theorem, the authors

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described all local solutions ∇ of the equation $\mathcal{R}ic^{\nabla} = 0$ with unknown real analytical connection ∇ on a real analytical manifold M. In [9], using the Cauchy–Kowalevski theorem, the authors described all local solutions ∇ of the equation $\mathcal{R}ic^{\nabla} = r$ with unknown real analytical (torsion free or not torsion free) connection ∇ on a real analytical manifold M, where r is a real analytical (not necessarily symmetric) tensor field of type $(0,2)$ on M.

2. The holomorphic version of the Cauchy–Kowalevski theorem. We will use the notation $(f)_k := \frac{\partial f}{\partial z^k}$ for a complex valued function f on a domain endowed with a holomorphic coordinate system $z^1 = x^1 + iy^1, \dots, z^n = x^n + iy^n$, where $\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right)$, $k = 1, \dots, n$.

The holomorphic version of the Cauchy–Kowalevski theorem can be found in [4]. We need only the following roughly presented particular case of it.

Theorem 1. *Suppose we have a system of differential equations*

$$(U^1)_1 = H^1\left(z^1,\dots,z^n,U^1,\dots,U^N,(U^1)_2,\dots,(U^1)_n,\dots,(U^N)_2,\dots,(U^N)_n\right),$$
$$(U^2)_1 = H^2\left(z^1,\dots,z^n,U^1,\dots,U^N,(U^1)_2,\dots,(U^1)_n,\dots,(U^N)_2,\dots,(U^N)_n\right),$$
$$\vdots$$
$$(U^N)_1 = H^N\left(z^1,\dots,z^n,U^1,\dots,U^N,(U^1)_2,\dots,(U^1)_n,\dots,(U^N)_2,\dots,(U^N)_n\right)$$

with unknown $\mathbb{C}$-valued functions $U^1, \dots, U^N$ in a neighborhood of $0 \in \mathbb{C}^n$, where H^i, $i = 1, \dots, N$ are holomorphic functions defined on some neighborhood of $(0, \dots, 0, \varphi^1(0), \dots, \varphi^N(0), (\varphi^1)_2(0), \dots, (\varphi^1)_n(0), \dots, (\varphi^N)_2(0), \dots, (\varphi^N)_n(0)) \in \mathbb{C}^{(N+1)n}$ for holomorphic functions $\varphi^1, \dots, \varphi^N$ given in a neighborhood of $0 \in \mathbb{C}^{n-1}$.

Then the system has a unique solution $(U^1(z^1,\dots,z^n),\dots,U^N(z^1,\dots,z^n))$ around $0 \in \mathbb{C}^n$ that is holomorphic and satisfies the initial conditions

$$U^i(0, z^2, \dots, z^n) = \varphi^i(z^2, \dots, z^n)$$

for $i = 1, \dots, N$.

3. Holomorphic connections. Consider a complex manifold (M, J) with $\dim_{\mathbb{C}} M = n$.

Let $T^{1,0}M$ be the bundle of complex vectors of type $(1,0)$ on M, [7]. Denote by $\Gamma(T^{1,0}M)$ the space of smooth ($\mathcal{C}^\infty$) complex vector fields of type $(1,0)$ on M.

A semi-connection on M is a $\mathbb{C}$-bilinear map $\nabla : \Gamma(T^{1,0}M) \times \Gamma(T^{1,0}M) \to \Gamma(T^{1,0}M)$ such that

$$\begin{aligned} \nabla_{fW} Z &= f \nabla_W Z \\ \nabla_W fZ &= W(f)Z + f\nabla_W Z \end{aligned} \tag{2}$$

for any $W, Z \in \Gamma(T^{1,0}M)$ and $f \in \mathcal{C}^\infty(M, \mathbb{C})$, where $\nabla_W Z := \nabla(W, Z)$. A semi-connection ∇ on M is called a holomorphic connection on M if

$\nabla_V W$ is a (locally defined) holomorphic vector field on M for any (locally defined) holomorphic vector fields V and W on M.

Let ∇ be a semi-connection on M. The curvature of ∇ is a $\mathcal{C}^\infty(M,\mathbb{C})$-trilinear map $\mathcal{R}^\nabla\colon \Gamma(T^{1,0}M)\times\Gamma(T^{1,0}M)\ \times\Gamma(T^{1,0}M)\to\Gamma(T^{1,0}M)$ defined by

$$\mathcal{R}^\nabla(U,W)Z=\nabla_U\nabla_W Z-\nabla_W\nabla_U Z-\nabla_{[U,W]}Z\,. \tag{3}$$

It can be treated as the fibre-wise $\mathbb{C}$-trilinear map

$$\mathcal{R}^\nabla\colon T^{1,0}M\times_M T^{1,0}M\times_M T^{1,0}M\to T^{1,0}M\,.$$

The Ricci tensor of ∇ is a $\mathcal{C}^\infty(M,\mathbb{C})$-bilinear map $\mathcal{R}ic^\nabla\colon\Gamma(T^{1,0}M)\times\Gamma(T^{1,0}M)\to\mathcal{C}^\infty(M,\mathbb{C})$ defined by

$$\mathcal{R}ic^\nabla(W,Z)_z=tr_{\mathbb{C}}\left(T^{1,0}_zM\ni v\to\mathcal{R}(v,W_z)Z_z\in T^{1,0}_zM\right). \tag{4}$$

The torsion of ∇ is a $\mathcal{C}^\infty(M,\mathbb{C})$-bilinear map $T^\nabla\colon\Gamma(T^{1,0}M)\times\Gamma(T^{1,0}M)\to\Gamma(T^{1,0}M)$ defined by

$$T^\nabla(W,Z)=\nabla_W Z-\nabla_Z W-[W,Z]\,.$$

We say that ∇ is torsion-free if $T^\nabla=0$.

Suppose $z^1=x^1+iy^1,\dots,z^n=x^n+iy^n$ is a holomorphic coordinate system defined on some open subset $\mathcal{U}\subset M$ and let $\frac{\partial}{\partial z^k}=\frac{1}{2}\left(\frac{\partial}{\partial x^k}-i\frac{\partial}{\partial y^k}\right),\ k=1,\dots,n$.

We write

$$\nabla_{\frac{\partial}{\partial z^j}}\frac{\partial}{\partial z^k}=\sum_{i=1}^n\Gamma^i_{jk}\frac{\partial}{\partial z^i} \tag{5}$$

for some uniquely determined $\mathbb{C}$-valued $\mathcal{C}^\infty$ maps $\Gamma^i_{jk},\ i,j,k=1,\dots,n$. If ∇ is holomorphic, then so are Γ^i_{jk}. The maps Γ^i_{jk} are called the Christoffel symbols of ∇ in the coordinates $z^1,\dots,z^n$.

For the Ricci tensor of ∇ we have

$$\mathcal{R}ic^\nabla\left(\frac{\partial}{\partial z^i},\frac{\partial}{\partial z^j}\right)=\sum_{k=1}^n\left[(\Gamma^k_{ij})_k-(\Gamma^k_{kj})_i\right]+\sum_{k,l=1}^n\left[\Gamma^l_{ij}\Gamma^k_{kl}-\Gamma^l_{kj}\Gamma^k_{il}\right]. \tag{6}$$

For the torsion of ∇ we have

$$T^\nabla\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z^j}\right)=\sum_{k=1}^n(\Gamma^k_{ij}-\Gamma^k_{ji})\frac{\partial}{\partial z^k}\,,\ i,j=1,\dots,n\,. \tag{7}$$

Clearly, ∇ is torsion-free if and only if its Christoffel symbols Γ^k_{ij} in any coordinates $z^1, \dots, z^n$ as above are symmetric in lower indices i and j.

Given a system of $\mathcal{C}^\infty$-maps $\theta^k_{ij} : \mathcal{U} \to \mathbb{C}$, $i, j, k = 1, \dots, n$, there exists a unique semi-connection θ on $\mathcal{U}$ with the Christoffel symbols θ^k_{ij}. If the maps θ^k_{ij} are holomorphic, then so is θ. If the maps θ^k_{ij} are symmetric in lower indices i and j, then θ is torsion-free.

4. How many holomorphic connections are there with a prescribed Ricci tensor? Let (M, J) be a complex manifold with $\dim_{\mathbb{C}} M = n$ and $z_o \in M$ be a point. Let $z^1 = x^1 + iy^1, \dots, z^n = x^n + iy^n$ be a holomorphic coordinate system defined on some neighborhood $\mathcal{U}$ of $z_o \in M$ with centrum z_o (i.e. z_o is 0 in these coordinates). For simplicity and without loss of generality we may assume that $M = \mathcal{U} \subset \mathbb{C}^n$ is an open neighborhood of $z_o = 0 \in \mathbb{C}^n$ and $z^1 = x^1 + iy^1, \dots, z^n = x^n + iy^n$ are the usual holomorphic coordinates.

Suppose r is a holomorphic tensor field of type $(0, 2)$ on M. It means that $r\colon \Gamma(T^{1,0}M) \times \Gamma(T^{1,0}M) \to \mathcal{C}^\infty(M, \mathbb{C})$ is a $\mathcal{C}^\infty(M, \mathbb{C})$-bilinear map such that $r(U, W)$ is holomorphic for any (locally defined) holomorphic vector fields U and W on M. We show how many (locally defined) holomorphic connections ∇ exist such that $\mathcal{R}ic^\nabla = r$ around z_o.

Set

$$r_{ij} = r\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right),\ i, j = 1, \dots, n\,.$$

Then $r_{ij} : \mathcal{U} \to \mathbb{C}$ are holomorphic. The condition (1) is equivalent to the system of partial differential equations

$$(8)\quad \sum_{k=1}^{n} \left[(\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i\right] = \sum_{k,l=1}^{n} \left[\Gamma^l_{kj}\Gamma^k_{il} - \Gamma^l_{ij}\Gamma^k_{kl}\right] + r_{ij}\,,\ i, j = 1, \dots, n\,.$$

Denote

$$(9)\qquad \Lambda_{ij} = \sum_{k,l=1}^{n} \left[\Gamma^l_{kj}\Gamma^k_{il} - \Gamma^l_{ij}\Gamma^k_{kl}\right].$$

Then we can rewrite (8) in the full form

$$(10)\qquad \left[(\Gamma^1_{ij})_1 + \ldots + (\Gamma^n_{ij})_n\right] - \left[(\Gamma^1_{1j})_i + \ldots + (\Gamma^n_{nj})_i\right] = \Lambda_{ij} + r_{ij}\,,$$

$i, j = 1, \dots, n$. The system (10) can be transformed equivalently to a system of the form as in the holomorphic version of the Cauchy–Kowalevski theorem. Namely, the system (10) can be transformed equivalently to the system

$$(11)\qquad \begin{aligned} &(\Gamma^n_{nj})_1 = -\Lambda_{1j} - r_{1j} + \Lambda'_{1j},\ j = 1, \dots, n \\ &(\Gamma^1_{ij})_1 = \Lambda_{ij} + r_{ij} - \Lambda'_{ij}\,,\ i = 2, \dots, n\,,\ j = 1, \dots, n\,, \end{aligned}$$

where

$$\Lambda'_{1j} = \left[(\Gamma^1_{1j})_1 + \ldots + (\Gamma^n_{1j})_n\right] - \left[(\Gamma^1_{1j})_1 + \ldots + (\Gamma^{n-1}_{n-1,j})_1\right], \; j = 1, \ldots, n$$

and

$$\Lambda'_{ij} = \left[(\Gamma^2_{ij})_2 + \ldots + (\Gamma^n_{ij})_n\right] - \left[(\Gamma^1_{1j})_i + \ldots + (\Gamma^n_{nj})_i\right],$$

$i = 2, \ldots, n$, $j = 1, \ldots, n$.

Indeed, for $i = 1$ and $j = 1, \ldots, n$, we keep each derivative $(\Gamma^n_{nj})_1$ on the left-hand side of the corresponding equation from the system (10). We denote the sum of all remaining terms on the left-hand side of the corresponding equation by Λ'_{1j} and move it to the right-hand side. Similarly, for $i > 1$ and $j = 1, \ldots, n$, we keep each derivative $(\Gamma^1_{ij})_1$ on the left-hand side of the corresponding equation from the system (10). We denote the sum of all remaining terms on the left-hand side of the corresponding equation by Λ'_{ij} and move it to the right-hand side. In this way we obtain the system (11) being equivalent to (10). We can see that the first derivatives which are on the left-hand sides of the system (11) are not presented in any terms Λ'_{ij} on the right-hand sides.

We are now in position to prove the following holomorphic version of Theorem 3.1 from [9].

Theorem 2. *Let (M, J) be a complex manifold and $\dim_{\mathbb{C}} M = n$ and $z_o \in M$ be a point. Let $n \geq 2$ and suppose $r\colon \Gamma(T^{1,0}M) \times \Gamma(T^{1,0}M) \to \mathcal{C}^\infty(M, \mathbb{C})$ is a holomorphic tensor field of type $(0, 2)$ on M (around z_o). There exists a locally defined around z_o holomorphic connection ∇ with $\mathcal{R}ic^\nabla = r$ around z_o. Moreover, the family of all locally defined around z_o holomorphic connections ∇ with $\mathcal{R}ic^\nabla = r$ around z_o depends bijectively on $n^3 - n^2$ holomorphic functions of n variables and n^2 holomorphic functions of $n - 1$ variables.*

Proof. We have $n + (n-1)n = n^2$ Christoffel symbols Γ^k_{ij} on the left-hand side of (11). Therefore, we have $n^3 - n^2$ Christoffel symbols Γ^k_{ij} not presented on the left-hand side of (11). We can choose them as arbitrary holomorphic functions. We can also choose n^2 arbitrary holomorphic functions of $n - 1$ variables to be the initial conditions. Then we can solve (11) by using the holomorphic version of the Cauchy–Kowalevski theorem and obtain the rest n^2 Christoffel symbols. □

5. How many torsion-free holomorphic connections are there with a prescribed Ricci tensor? Let (M, J) be a complex manifold, $z_o \in M$ be a point and $\mathcal{U}$ be an open neighborhood of z_o and suppose $z^1, \ldots, z^n$ are holomorphic coordinates on $\mathcal{U}$ with centrum z_o (i.e. z_o is 0 in these coordinates). For simplicity and without loss of generality we may assume that $M = \mathcal{U} \subset \mathbb{C}^n$ is an open neighborhood of $z_o = 0 \in \mathbb{C}^n$ and $z^1 = x^1 + iy^1, \ldots, z^n = x^n + iy^n$ are the usual holomorphic coordinates.

Consider a torsion-free holomorphic connection ∇ on $\mathcal{U}$ with the Christoffel symbols Γ^k_{ij} (with $\Gamma^k_{ij} = \Gamma^k_{ji}$) in the holomorphic coordinates $z^1, \dots, z^n$.

We can decompose the Ricci tensor of ∇ into its symmetric and antisymmetric parts. We obtain $\mathcal{R}ic^{\nabla} = s + a$, where

$$s(W,Z) = \frac{\mathcal{R}ic^{\nabla}(W,Z) + \mathcal{R}ic^{\nabla}(Z,W)}{2}\,, \tag{12}$$

$$a(W,Z) = \frac{\mathcal{R}ic^{\nabla}(W,Z) - \mathcal{R}ic^{\nabla}(Z,W)}{2}\,. \tag{13}$$

By (6) we have

$$\mathcal{R}ic^{\nabla}\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = \sum_{k=1}^{n} (\Gamma^k_{ij})_k - (D_j)_i - \Lambda_{ij}\,,\ i,j = 1, \dots, n\,, \tag{14}$$

where Λ_{ij} are given in (9) and

$$D_j = \sum_{k=1}^{n} \Gamma^k_{kj}\,,\ j = 1, \dots, n\,. \tag{15}$$

Since ∇ is torsion-free, the portions $\sum\limits_{k=1}^{n} (\Gamma^k_{ij})_k$ and Λ_{ij} are symmetric for i and j. Hence

$$a_{ij} = a\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = \frac{(D_i)_j - (D_j)_i}{2}\,,\ i,j = 1, \dots, n\,, \tag{16}$$

$$s_{ij} = s\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = \sum_{k=1}^{n} (\Gamma^k_{ij})_k - \frac{(D_j)_i + (D_i)_j}{2} - \Lambda_{ij}\,,\ i,j = 1, \dots, n\,.$$

Moreover, since ∇ is holomorphic, then a is a holomorphic 2-form, i.e. $\overline{\partial}$-closed form of degree $(2,0)$, where $d = \partial + \overline{\partial}$ is the well-known decomposition.

Proposition 1. *The antisymmetric part a of the Ricci tensor $\mathcal{R}ic^{\nabla}$ of a torsion-free holomorphic connection ∇ on a complex manifold M is ∂-closed.*

Proof. Let $\alpha = \sum_{i=1}^{n} D_i dz^i$. The equation (16) can be equivalently rewritten into $a = \partial\alpha$ on $\mathcal{U}$. Then $\partial a = \partial\partial\alpha = 0$ on $\mathcal{U}$. □

We are now in position to prove the following holomorphic version of Theorem 3.4 from [9].

Theorem 3. *Let (M, J) be a complex manifold with $\dim_{\mathbb{C}} M = n$ and $z_o \in M$. Let $n \geq 2$. Let $r\colon \Gamma(T^{1,0}M) \times \Gamma(T^{1,0}M) \to \mathcal{C}^{\infty}(M, \mathbb{C})$ be a defined around z_o holomorphic tensor field of type $(0,2)$ on M such that the (treated as holomorphic 2-form) antisymmetric part a of r is ∂-closed. The family of locally defined around z_o torsion-free holomorphic connections ∇ with*

$\mathcal{R}ic^\nabla = r$ depends bijectively on $\frac{n^3-3n}{2}$ holomorphic functions of n variables and $\frac{n^2+n}{2}$ holomorphic functions of $n-1$ variables and one holomorphic function vanishing in z_o of n variables. In particular, there exists a locally defined around z_o torsion-free holomorphic connection ∇ with $\mathcal{R}ic^\nabla = r$ around z_o.

Proof. Without loss of generality we may assume that $M = \mathcal{U} \subset \mathbb{C}^n$ is an open neighborhood of $z_o = 0 \in \mathbb{C}^n$ and $z^1 = x^1 + iy^1, \dots, z^n = x^n + iy^n$ are the usual holomorphic coordinates. Let s be the symmetric part of r and a be the antisymmetric one. Let

$$a_{ij} = a\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right)\ ,\ s_{ij} = s\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right),\ r_{ij} = s_{ij} + a_{ij}\,.$$

By the (holomorphic) Poincare lemma, we (restricting $\mathcal{U}$ if necessary) may write $a = \partial\alpha$ for some holomorphic 1-form α on $\mathcal{U}$. Such α is unique modulo $\partial\varphi$, where φ is a holomorphic map of n variables with $\varphi(z_o) = 0$. We can write $\alpha = \sum_{i=1}^n D_i dz^i$ for some holomorphic maps $D_i : \mathcal{U} \to \mathbb{C}$.

So, we have given holomorphic functions r_{ij}, s_{ij}, a_{ij}, D_i satisfying

$$a_{ij} = \frac{(D_i)_j - (D_j)_i}{2}\ ,\ i,j = 1,\dots,n\,, \tag{17}$$

and then

$$\frac{(D_i)_j + (D_j)_i}{2} = a_{ij} + (D_j)_i \tag{18}$$

for $i, j = 1, \dots, n$.

Clearly, if a torsion-free holomorphic connection ∇ with the Christoffel symbols Γ^i_{jk} with $\Gamma^i_{jk} = \Gamma^i_{kj}$ satisfies the conditions

$$D_j = \sum_{k=1}^n \Gamma^k_{kj}\ ,\ j = 1,\dots,n\,, \tag{19}$$

$$a_{ij} = \frac{(D_i)_j - (D_j)_i}{2}\ ,\ i,j = 1,\dots,n\,, \tag{20}$$

$$s_{ij} = \sum_{k=1}^n (\Gamma^k_{ij})_k - \frac{(D_j)_i + (D_i)_j}{2} - \Lambda_{ij}\ ,\ i,j = 1,\dots,n\,, \tag{21}$$

where Λ_{ij} is as in (9), then

$$r_{ij} = a_{ij} + s_{ij} = \sum_{k=1}^n (\Gamma^k_{ij})_k - (D_j)_i - \Lambda_{ij} = \mathcal{R}ic^\nabla\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right), \tag{22}$$

$i, j = 1, \dots, n$, see (14), and then $\mathcal{R}ic^\nabla = r$.

Conversely, if $\mathcal{R}ic^\nabla = r$, then the above system of systems (19) and (20) and (21) holds possibly with another D_j's, see the beginning of this section.

Of course, the another D_j's are of the form $D_j + (\varphi)_j$ for some holomorphic function φ (in n variables) vanishing in z_o, see above.

So, it remains to describe all local (near z_o) holomorphic solutions of the system consisting of equations (19) for $j = 1, \dots, n$ and (20) for $i, j = 1, \dots, n$ and (21) for $i, j = 1, \dots, n$ (with unknown torsion-free holomorphic connection ∇ with Christoffel symbols Γ^i_{jk} with $\Gamma^i_{jk} = \Gamma^i_{kj}$).

But the conditions (20) for $i, j = 1, \dots, n$ are satisfied because of (17).

Further, because of the symmetry in indices i and j, the system of equations (21) for $i, j = 1, \dots, n$ is equivalent to the one for i and j with $1 \le i \le j \le n$, i.e. to the system

$$s_{ij} = \sum_{k=1}^{n} (\Gamma^k_{ij})_k - \frac{(D_j)_i + (D_i)_j}{2} - \Lambda_{ij}$$

for i and j with $1 \le i \le j \le n$, where Λ_{ij} are defined in (9), i.e. (because of (18)) to the system

$$s_{ij} = (\Gamma^1_{ij})_1 + (\Gamma^2_{ij})_2 + \dots + (\Gamma^n_{ij})_n - a_{ij} - (D_j)_i - \Lambda_{ij} \tag{23}$$

for i and j with $1 \le i \le j \le n$.

If $i = j = 1$, because of (19), the equation (23) can be rewritten (equivalently) as

$$(\Gamma^2_{12})_1 = \sum_{k=2}^{n} (\Gamma^k_{11})_k - \sum_{k=3}^{n} (\Gamma^k_{k1})_1 - \Lambda_{11} - r_{11}\,. \tag{24}$$

If $(i, j) \neq (1, 1)$, the equation (23) can be rewritten (equivalently) as

$$(\Gamma^1_{ij})_1 = -(\Gamma^2_{ij})_2 - \dots - (\Gamma^n_{ij})_n + \Lambda_{ij} + (D_j)_i + r_{ij}\,. \tag{25}$$

Consequently, we get the following equivalent to (21) system of $\frac{n(n+1)}{2}$ equations

$$\begin{aligned}
&(\Gamma^2_{12})_1 = \sum_{k=2}^{n} (\Gamma^k_{11})_k - \sum_{k=3}^{n} (\Gamma^k_{1k})_1 - \Lambda_{11} - r_{11}\,,\\
&(\Gamma^1_{1j})_1 = -(\Gamma^2_{1j})_2 - \dots - (\Gamma^n_{1j})_n + \Lambda_{1j} + (D_1)_j + r_{j1}\,,\ j > 1\,,\\
&(\Gamma^1_{ij})_1 = -(\Gamma^2_{ij})_2 - \dots - (\Gamma^n_{ij})_n + \Lambda_{ij} + (D_j)_i + r_{ij}\,,\ 1 < i \le j \le n\,.
\end{aligned} \tag{26}$$

Additionally, the Christoffel symbols satisfy (19), i.e.

$$\begin{aligned}
D_1 &= \Gamma^1_{11} + \Gamma^2_{21} + \dots + \Gamma^n_{n1}\,,\\
D_2 &= \Gamma^1_{12} + \Gamma^2_{22} + \dots + \Gamma^n_{n2}\,,\\
&\vdots\\
D_n &= \Gamma^1_{1n} + \Gamma^2_{2n} + \dots + \Gamma^n_{nn}\,.
\end{aligned} \tag{27}$$

It remains to describe all local (near z_o) holomorphic solutions of the above system of systems (26) and (27) with unknown Christoffel symbols $\Gamma^i_{jk} = \Gamma^i_{kj}$, where (of course) the quantities r_{11}, D_j, $(D_1)_j + r_{j1}$ and $(D_j)_i + r_{ij}$ are the given ones.

We can see that on the right-hand sides of (27) there are no repeated Christoffel symbols. From the first equation of (27) we determine $\Gamma^1_{11} = D_1 - \Gamma^2_{21} - \ldots - \Gamma^n_{n1}$ and then insert it into (26). From the next equations we determine Γ^k_{kk} and then insert it into (26). Then the derivatives from the left-hand side of (26) (after the substitutions) will not appear on the right-hand side of (26) (after the substitutions). So, for the modified system (26) (after the substitutions) we can apply the holomorphic version of the Cauchy–Kowalevski theorem.

The number of holomorphic functions of n variables we can choose arbitrarily is the number of all Christoffel symbols minus the number of equations of (26) (after the substitutions) minus the number of equations of (27), i.e. $n\frac{(n+1)n}{2} - \frac{n(n+1)}{2} - n = \frac{n^3-3n}{2}$. The number of holomorphic functions of $n-1$ variables we can choose arbitrarily is the number of equations of (26) (after the substitutions) i.e. $\frac{n^2+n}{2}$. □

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