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# Singular linear $q$-Hamiltonian systems

Abstract. In this paper, a singular linear $q$-Hamiltonian system is considered. The Titchmarsh–Weyl theory for this problem is constructed. Firstly, we provide some necessary fundamental concepts of the $q$-calculus. Later, we studied Titchmarsh–Weyl functions $M(\lambda)$ and circles $\mathcal{C}_{TW}(a,\lambda)$ for this system. Circles $\mathcal{C}_{TW}(a,\lambda)$ are proved to be nested. In the fourth part of the work, the number of square-integrable solutions of this system is studied. In the fifth part of the work, boundary conditions in the singular case are obtained. Finally, a self-adjoint operator is defined.

**1. Introduction.** It is well known that Hamiltonian systems have played an important role in modeling various physical systems, for example, in the study of electromechanical, electrical, and complex network systems with negligible dissipation (see [17]). The theory of Hamiltonian systems is well developed (see ([2, 5, 7, 9, 10, 11, 12, 14, 15, 16]). Recently, different approaches to Hamiltonian systems have been introduced. One of these approaches is the quantum calculus approach. In other words, it is the application of quantum calculus to Hamiltonian systems (see [1, 6]). This paper is a continuation of [1], where the authors investigated the following systems

$$Jy^{[q]}(\zeta) = [\lambda V(\zeta) + T(\zeta)]\, y(\zeta)\,,\ \zeta \in (0,a)\,,\ a > 0,$$

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where

$$y^{[q]}(\zeta)=\begin{pmatrix} D_q y_1(\zeta) \\ \frac{1}{q}D_{q^{-1}}y_2(\zeta)\end{pmatrix},\ V(\zeta)=\begin{pmatrix} V_1(\zeta) & O_n \\ O_n & V_2(\zeta)\end{pmatrix},$$

$$J=\begin{pmatrix} O_n & -I_n \\ I_n & O_n\end{pmatrix},\qquad T(\zeta)=\begin{pmatrix} T_1(\zeta) & T_2^*(\zeta) \\ T_2(\zeta) & T_3(\zeta)\end{pmatrix}$$

and $V(.)$, $T(.)$ are $2n\times 2n$ complex Hermitian matrix-valued functions defined on $[0,a]$ and continuous at zero.

In [2], the authors studied the following discontinuous linear Hamiltonian system:

$$J\mathcal{Z}'(x)-B_2(x)\mathcal{Z}(x)=\lambda B_1(x)\mathcal{Z}(x),\ x\in[a,c)\cup(c,b),$$

where $-\infty<a<c<b\leq+\infty$, $\lambda\in\mathbb{C}$; $B_1(.)$ and $B_2(.)$ are $2n\times 2n$ complex Hermitian matrix-valued functions defined on $[a,c)\cup(c,b)$ whose entries are Lebesgue measurable and locally integrable functions on $[a,c)\cup(c,b)$; $B_1(x)$ is nonnegative-definite and

$$J=\begin{pmatrix} 0 & -I_n \\ I_n & 0\end{pmatrix},$$

where $I_n$ is the identity matrix in $\mathbb{C}^n$. For these systems, Titchmarsh–Weyl functions and circles are investigated. The number of square-integrable solutions of this system is studied.

In [3], Allahverdiev and Tuna studied the following singular Hahn–Hamiltonian systems

$$J\mathcal{Z}^{[h]}(x)-B(x)\mathcal{Z}(x)=\lambda W(x)\mathcal{Z}(x),\ x\in[\omega_0,\infty),$$

where the matrices

$$B(x)=\begin{pmatrix} B_1(x) & B_2^*(x) \\ B_2(x) & B_3(x)\end{pmatrix}$$

and $W(.)$ are $2n\times 2n$ complex Hermitian matrix-valued functions defined on $[\omega_0,\infty)$ which are continuous at $\omega_0$; $\mathcal{Z}(x)$ is $2n\times 1$ vector-valued function;

$$\mathcal{Z}^{[h]}(x)=\begin{pmatrix} D_{\omega,q}\mathcal{Z}_1(x) \\ \frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{Z}_2(x)\end{pmatrix}=\begin{pmatrix} D_{\omega,q}\mathcal{Z}_1(x) \\ \frac{1}{q}D_{\omega,q}\mathcal{Z}_2\left(h^{-1}(x)\right)\end{pmatrix}$$

and

$$J=\begin{pmatrix} 0 & -I_n \\ I_n & 0\end{pmatrix},$$

where $I_n$ is the $n\times n$ identity matrix. To pass from the Hahn difference expression to operators, they defined the Hilbert space $L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n})$. The corresponding maximal operator $L_{\max}$ was introduced. For this system, they proved Green's formula. Later, a regular, self-adjoint Hahn–Hamiltonian system was introduced. Titchmarsh–Weyl functions $M(\lambda)$ and circles $\mathcal{C}(a,\lambda)$ for this system were studied. It was proved that these circles are nested. The number of square-integrable solutions of the Hahn–Hamilton system was studied. Finally, a self-adjoint operator was obtained.

In this study, the $q$-Hamiltonian system in the singular case is discussed based on the studies mentioned above. For this system, the Titchmarsh–Weyl function $M(\lambda)$ and circles $\mathcal{C}_{TW}(a,\lambda)$ are investigated. The circles $\mathcal{C}_{TW}(a,\lambda)$ are proved to be nested. The number of square-integrable solutions of this system is studied. Boundary conditions in the singular case are obtained. Finally, a self-adjoint operator is defined. To the best of authors knowledge such study has not been reported earlier in the literature. In the analysis that follows, we will largely follow the development of the theory in [2, 3, 14, 15].

**2. Preliminaries.** In this section, the basic concepts of $q$-calculus that will be used in the article are given. For more detailed information, the following sources can be examined: [4, 8, 13].

From now on, $\zeta$ denotes the independent variable and $q \in (0,1)$. Let $A \subset \mathbb{R}$ be a $q$-geometric set, i.e., if $q\zeta \in A$ for all $\zeta \in A$. We begin by defining the *operator* $\mathcal{D}_q$ by

$$\mathrm{D}_q\varphi(x) = \begin{cases} \frac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{if } q \in \mathbb{R}\backslash\{1\},\ x \neq 0; \\ \frac{d\varphi(x)}{dx} & \text{if } q = 1; \\ \frac{d\varphi(x)}{dx} & \text{if } x = 0. \end{cases}$$

where $\zeta \in A$ (see [8]). We define the *Jackson q-integration* by

$$\int_0^\zeta f(\gamma)\, d_q\gamma = \zeta(1-q)\sum_{n=0}^{\infty} q^n f(q^n\zeta),\ \zeta \in A,$$

provided that the series converges and

$$\int_a^b f(\gamma)\, d_q\gamma = \int_0^b f(\gamma)\, d_q\gamma - \int_0^a f(\gamma)\, d_q\gamma,$$

where $a, b \in A$. Through the remainder of the paper, we deal only with functions $q$-regular at zero, i.e, functions satisfying

$$\lim_{n\to\infty} f(\zeta q^n) = f(0)$$

for every $\zeta \in A$.

**3. Singular system.** Let us consider the following system:

(1) $$\Gamma(\mathcal{Z}) := J\mathcal{Z}^{[q]}(\zeta) - A(\zeta)\mathcal{Z}(\zeta) = \lambda B(\zeta)\mathcal{Z}(\zeta),\ \zeta \in (0,\infty),\ \lambda \in \mathbb{C},$$

where

$$A(.) = \begin{pmatrix} A_1(.) & A_2^*(.) \\ A_2(.) & A_3(.) \end{pmatrix}$$

and $B(.)$ are $2n\times 2n$ complex Hermitian matrix-valued functions, defined on $[0,\infty)$ and continuous at zero, $B(.)$ is non-negative definite; $I+(q-1)\zeta A_2(\zeta)$ is invertible; $\mathcal{Z}(\zeta)$ is $2n\times 1$ vector-valued function

$$\mathcal{Z}(\zeta)=\begin{pmatrix}\mathcal{Z}_1(\zeta)\\ \mathcal{Z}_2(\zeta)\end{pmatrix},\ \mathcal{Z}^{[q]}(\zeta)=\begin{pmatrix}D_q\mathcal{Z}_1(\zeta)\\ \frac{1}{q}D_{q^{-1}}\mathcal{Z}_2(\zeta)\end{pmatrix},$$

$$J=\begin{pmatrix}0 & -I_n\\ I_n & 0\end{pmatrix},$$

where $I_n$ is the identity matrix in $\mathbb{C}^n$.

Let

$$L^2_{q,B}((0,\infty);\mathbb{C}^{2n}):=\left\{\mathcal{Z}:\int_0^\infty(B\mathcal{Z},\mathcal{Z})_{\mathbb{C}^{2n}}\,d_q\zeta<\infty\right\} \tag{2}$$

be the $2n$-dimensional Hilbert space vector-valued functions with the inner product

$$(\mathcal{Z},\mathcal{Y}):=\int_0^\infty(B\mathcal{Z},\mathcal{Y})_{\mathbb{C}^{2n}}\,d_q\zeta=\int_0^\infty\mathcal{Y}^*(\zeta)B(\zeta)\mathcal{Z}(\zeta)d_q\zeta.$$

We will need the following assumptions:

1) if $\Gamma(\mathcal{Z})=BF$ and $B\mathcal{Z}=0$, then $\mathcal{Z}=0$.

2) $\int_0^\infty\mathcal{Z}^*(\zeta)B(\zeta)\mathcal{Z}(\zeta)d_q\zeta>0$, where $\mathcal{Z}$ is a nontrivial solution of (1).

Let $\eta_1,\eta_2,\xi_1,\xi_2$ be matrices satisfying

$$\eta_1\eta_1^*+\eta_2\eta_2^*=I_n,\ \eta_1\eta_2^*-\eta_2\eta_1^*=0, \tag{3}$$

$$\xi_1\xi_1^*+\xi_2\xi_2^*=I_n,\ \xi_1\xi_2^*-\xi_2\xi_1^*=0, \tag{4}$$

and $\operatorname{rank}\begin{pmatrix}\eta_1 & \eta_2\end{pmatrix}=\operatorname{rank}\begin{pmatrix}\xi_1 & \xi_2\end{pmatrix}=n$.

Next, we will assume the following conditions:

$$\Omega\widehat{\mathcal{Z}}(0)=0, \tag{5}$$

$$\Xi\widehat{\mathcal{Z}}(a)=0, \tag{6}$$

where $0<a<\infty$,

$$\widehat{\mathcal{Z}}(\zeta)=\begin{pmatrix}\mathcal{Z}_1(\zeta)\\ \mathcal{Z}_2(q^{-1}\zeta)\end{pmatrix},$$

and

$$\Omega=\begin{pmatrix}\eta_1 & \eta_2\\ 0 & 0\end{pmatrix},\ \Xi=\begin{pmatrix}0 & 0\\ \xi_1 & \xi_2\end{pmatrix}.$$

From (5), we get $\Omega J\Omega^*=0$ and $\Xi J\Xi^*=0$. Eq. (1) with conditions (5), (6) defines a regular, self-adjoint problem (see [1]).

Let

$$Z=\begin{pmatrix}\varphi & \psi\end{pmatrix}=\begin{pmatrix}\varphi_1 & \psi_1\\ \varphi_2 & \psi_2\end{pmatrix} \tag{7}$$

be the fundamental matrix for $\Gamma(\mathcal{Z}) = \lambda B\mathcal{Z}$ satisfying

$$\widehat{Z}(0) = E := \begin{pmatrix} \eta_1^* & -\eta_2^* \\ \eta_2^* & \eta_1^* \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \eta_1 & \eta_2 \end{pmatrix} \widehat{\varphi}(0) = I_n$$

and

$$\begin{pmatrix} \eta_1 & \eta_2 \end{pmatrix} \widehat{\psi}(0) = 0.$$

**Lemma 1.** *The following relation holds*

$$\widehat{Z}^*(\zeta,\lambda) J\widehat{Z}(\zeta,\lambda) = J. \tag{8}$$

**Proof.** From Green's formula (see [1]), we conclude that

$$\begin{aligned} 0 &= \int_0^\zeta \widehat{Z}^*(t,\lambda) B(t)\Gamma(Z(t,\lambda))d_qt - \int_0^\zeta \Gamma(Z^*(t,\lambda))B(t)Z(t,\lambda)d_qt \\ &= \widehat{Z}^*(\zeta,\lambda) J\widehat{Z}(\zeta,\lambda) - \widehat{Z}^*(0,\lambda) J\widehat{Z}(0,\lambda). \end{aligned}$$

Thus

$$\widehat{Z}^*(\zeta,\lambda) J\widehat{Z}(\zeta,\lambda) = \widehat{Z}^*(0,\lambda) J\widehat{Z}(0,\lambda).$$

Since $\widehat{Z}(0,\lambda) = E$, we obtain $\widehat{Z}^*(\zeta,\lambda) J\widehat{Z}(\zeta,\lambda) = J$. □

**4. Titchmarsh–Weyl function.** In this section, we introduce the Titchmarsh–Weyl matrix-valued function $M(\lambda)$ for the system (1), (5).

**Definition 2.** Let

$$\widehat{Y}_a(\zeta,\lambda) = \widehat{Z}(\zeta,\lambda) \begin{pmatrix} I_n \\ M(a,\lambda) \end{pmatrix},$$

where $\operatorname{Im}\lambda \neq 0$ and $M(a,\lambda)$ is an $n \times n$ matrix-valued function.

**Theorem 3.** *Let*

$$\begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} \widehat{Y}_a(a,\lambda) = 0. \tag{9}$$

*Then, we obtain*

$$M(a,\lambda) = -\left(\xi_1\psi_1(a) + \xi_2\psi_2\left(q^{-1}a\right)\right)^{-1}\left(\xi_1\varphi_1(a) + \xi_2\varphi_2\left(q^{-1}a\right)\right)$$

*and*

$$\widehat{Y}_a^*(a,\lambda) J\widehat{Y}_a(a,\lambda) = 0,$$

*where $\xi_1$ and $\xi_2$ are defined in* (4).

*Conversely, if $\widehat{Y}_a$ satisfies*

$$\widehat{Y}_a^*(a,\lambda) J\widehat{Y}_a(a,\lambda) = 0,$$

*then there exist $\xi_1, \xi_2$ satisfying* (4) *such that*

$$\begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} \widehat{Y}_a(a,\lambda) = 0$$

*and*

$$M(a,\lambda)=-\left(\xi_1\psi_1(a)+\xi_2\psi_2\left(q^{-1}a\right)\right)^{-1}\left(\xi_1\varphi_1(a)+\xi_2\varphi_2\left(q^{-1}a\right)\right).$$

**Proof.** Let

$$\begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix}\widehat{Y}_a(a,\lambda)=0.$$

Then we get

$$\left[\xi_1\psi_1(a)+\xi_2\psi_2\left(q^{-1}a\right)\right]M(a,\lambda)=-\left(\xi_1\varphi_1(a)+\xi_2\varphi_2\left(q^{-1}a\right)\right),$$

and

$$M(a,\lambda)=-\left(\xi_1\psi_1(a)+\xi_2\psi_2\left(q^{-1}a\right)\right)^{-1}\left(\xi_1\varphi_1(a)+\xi_2\varphi_2\left(q^{-1}a\right)\right).$$

The inverse of the matrix $\xi_1\psi_1(a)+\xi_2\psi_2\left(q^{-1}a\right)$ exists because $\lambda$ is not an eigenvalue. From (9) we deduce that

$$\widehat{Y}_a(a,\lambda)=\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix}K,$$

for

$$\begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix}\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix}K=0.$$

Thus, we have

$$\begin{pmatrix} I_n & M^*(a,\lambda) \end{pmatrix}\widehat{Z}^*(a,\lambda)J\widehat{Z}(a,\lambda)\begin{pmatrix} I_n \\ M(a,\lambda) \end{pmatrix}=0,$$

i.e.,

$$\widehat{Y}_a^*(a,\lambda)J\widehat{Y}_a(a,\lambda)=0.$$

Conversely, for some $M$, let

$$\begin{aligned}&\widehat{Y}_a^*(a,\lambda)J\widehat{Y}_a(a,\lambda)\\&=\begin{pmatrix} I_n & M^*(a,\lambda) \end{pmatrix}\widehat{Z}^*(a,\lambda)J\widehat{Z}(a,\lambda)\begin{pmatrix} I_n \\ M(a,\lambda) \end{pmatrix}=0.\end{aligned}$$

We get the claim by letting

$$\begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix}=\begin{pmatrix} I_n & M^*(a,\lambda) \end{pmatrix}\widehat{Z}^*(a,\lambda)J.$$

□

**Definition 4.** Let

$$\mathcal{C}_{TW}(a,\lambda)=\begin{pmatrix} I_n & M^*(a,\lambda) \end{pmatrix}\begin{pmatrix} C_1 & C_2^* \\ C_2 & C_3 \end{pmatrix}\begin{pmatrix} I_n \\ M(a,\lambda) \end{pmatrix}=0, \tag{10}$$

where $C_m$ are $n\times n$ matrices for $m=1,2,3$ and

$$\begin{pmatrix} C_1 & C_2^* \\ C_2 & C_3 \end{pmatrix}=-\operatorname{sgn}(\operatorname{Im}\lambda)\,\widehat{Z}^*\left(a,\overline{\lambda}\right)(J/i)\,\widehat{Z}(a,\lambda). \tag{11}$$

From Definition 4, we conclude that

$$\begin{aligned}\mathcal{C}_{TW}(a,\lambda) &= \left(M_a + C_3^{-1}C_2\right)^* C_4 \left(M_a + C_3^{-1}C_2\right) + C_1 - C_2^* C_3^{-1} C_2 \\ &= (M_a - C_4) K_1^{-2} (M_a - C_4) - K_2^2 = 0,\end{aligned}$$

where $C_4 = -C_3^{-1}C_2$, $K_1^{-2} = C_3^{-1}$ and $K_2^2 = C_2^* C_3^{-1} C_2 - C_1$.

**Lemma 5.** *We have $C_3 > 0$.*

**Proof.** From (7) and (11), we see that

$$\begin{aligned}\begin{pmatrix} C_1 & C_2^* \\ C_2 & C_3 \end{pmatrix} &= -\operatorname{sgn}(\operatorname{Im}\lambda) \begin{pmatrix} \varphi_1^*(\zeta) & \varphi_2^*\left(q^{-1}\zeta\right) \\ \psi_1^*(\zeta) & \psi_2^*\left(q^{-1}\zeta\right) \end{pmatrix} \\ &\quad \times \begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(\zeta) & \psi_1(\zeta) \\ \varphi_2\left(q^{-1}\zeta\right) & \psi_2\left(q^{-1}\zeta\right) \end{pmatrix} \\ &= -\operatorname{sgn}(\operatorname{Im}\lambda) \begin{pmatrix} \widehat{\varphi}^*(J/i)\widehat{\varphi} & \widehat{\varphi}^*(J/i)\widehat{\psi} \\ i\widehat{\psi}^*(J/i)\widehat{\varphi} & \widehat{\psi}^*(J/i)\widehat{\psi} \end{pmatrix}.\end{aligned}$$

Hence

$$C_3 = -\operatorname{sgn}(\operatorname{Im}\lambda)\,\widehat{\psi}^*(J/i)\,\widehat{\psi}.$$

A direct calculation gives

$$2\operatorname{Im}\lambda\left(\int_0^a \psi^* B \psi d_q\zeta\right) = \widehat{\psi}^*(J/i)\,\widehat{\psi}(a) - \widehat{\psi}^*(J/i)\,\widehat{\psi}(0).$$

Since $\widehat{\psi}^*(J/i)\,\widehat{\psi}(0) = 0$, we get the desired result. □

**Lemma 6.** *We have $C_2^* C_3^{-1} C_2 - C_1 = \overline{C_3}^{-1} > 0$, where $\overline{C_3}^{-1} := C_3^{-1}\left(\overline{\lambda}\right)$.*

**Proof.** From (8) we have $\widehat{Z}(\zeta,\lambda)\,J\widehat{Z}^*(\zeta,\lambda) = J$. Hence

$$\begin{aligned}J &= \widehat{Z}^*\left(\zeta,\overline{\lambda}\right)\left[-J\widehat{Z}(\zeta,\lambda)\,J\widehat{Z}^*(\zeta,\lambda)\,J\right]\widehat{Z}\left(\zeta,\overline{\lambda}\right) \\ &= -\left[\widehat{Z}^*\left(\zeta,\overline{\lambda}\right)(J/i)\,\widehat{Z}(\zeta,\lambda)\right] J \left[-\widehat{Z}^*(\zeta,\lambda)\,(J/i)\,\widehat{Z}\left(\zeta,\overline{\lambda}\right)\right],\end{aligned}$$

so

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = -\begin{pmatrix} C_1 & C_2^* \\ C_2 & C_3 \end{pmatrix}\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} \overline{C_1} & \overline{C_2^*} \\ \overline{C_2} & \overline{C_3} \end{pmatrix},$$

since there is a sign change in the matrix when $\lambda$ replaces $\overline{\lambda}$. Consequently,

$$\begin{aligned} 0 &= C_1\overline{C_2} - C_2^*\overline{C_1}, & -I_n &= C_1\overline{C_3} - C_2^*\overline{C_2}, \\ I_n &= C_2\overline{C_2} - C_3\overline{C_1}, & 0 &= C_2\overline{C_3} - C_3\overline{C_2^*}.\end{aligned}$$

The last and second equation show that

$$\overline{C_3}^{-1} = C_2^* C_3^{-1} C_2 - C_1.$$

□

**Corollary 7.** *We have $K_2 = \overline{K_1}$.*

**Theorem 8.** *As $a$ increases, $C_3$, $K_1$ and $K_2$ decrease.*

**Proof.** The theorem follows from the equality

$$C_3 = 2\left|\operatorname{Im}\lambda\right| \left(\int_0^a \psi^* B\psi d_q\zeta\right).$$

□

**Corollary 9.** *The following relations hold:*

$$\lim_{a\to\infty} K_1(a,\lambda) = K_0, \quad \lim_{a\to\infty} K_2(a,\lambda) = \overline{K_0},$$

*where $K_0 \geq 0$ and $\overline{K_0} \geq 0$.*

**Theorem 10.** *As $a \to \infty$, the circles $\mathcal{C}_{TW}(a,\lambda) = 0$ are nested.*

**Proof.** The interior of the circle is

$$-\operatorname{sgn}(\operatorname{Im}\lambda) \begin{pmatrix} I_n & M^*(a,\lambda) \end{pmatrix} \widehat{Z}^*(a,\overline{\lambda})(J/i)\widehat{Z}(a,\lambda) \begin{pmatrix} I_n \\ M(a,\lambda) \end{pmatrix} \leq 0.$$

From (10) it follows that

$$\mathcal{C}_{TW}(a,\lambda) = 2\left|\operatorname{Im}\lambda\right| \left(\int_0^a Y_a^* BY_a d_q\zeta\right) \pm \frac{1}{i}(M_a^* - M_a).$$

Hence, if $M_a$ is in the circle at $a_2 > a$, then $\mathcal{C}_{TW}(a,\lambda) \leq 0$ at $a_2$. At $a_2$, $\mathcal{C}_{TW}(a,\lambda)$ is certainly smaller and so $\mathcal{C}_{TW}(a,\lambda)$ is in the circle at $a_2$ as well. As $a \to \infty$, $\mathcal{C}_{TW}(a,\lambda) = 0$ are nested. □

**Theorem 11.** *The following relation holds*

$$\lim_{a\to\infty} \mathcal{C}_{TW}(a,\lambda) = \mathcal{C}_{TW}^0.$$

**Proof.** By (10), we obtain

$$\mathcal{C}_{TW}(a,\lambda) = (M_a - D)^* K_1^{-2}(M_a - D) - K_2^2 = 0.$$

Hence

$$\left[K_1^{-1}(M_a - D)\overline{K_1^{-1}}\right]^* \left[K_1^{-1}(M_a - D)\overline{K_1^{-1}}\right] = I_n. \tag{12}$$

From (12) it follows that $U = K_1^{-1}(M_a - D)\overline{K_1^{-1}}$, where $U$ is a unitary matrix. Thus

$$M_a(\lambda) = D + K_1 U \overline{K_1}. \tag{13}$$

As $U$ varies over the $n \times n$ unit sphere, $M_a(\lambda)$ varies over a circle with the center $D$.

Let $D_1$ be the center at $a'$, $D_2$ be the center at $a''$. From (13) it follows that

$$M_{a'}(\lambda) = D_1 + K_1(a') U_1 \overline{K_1(a')}$$

and

$$M_{a''}(\lambda) = D_2 + K_1(a'') U_2 \overline{K_1(a'')}. \tag{14}$$

Since $\mathcal{C}_{TW}(a'', \lambda) \subset \mathcal{C}_{TW}(a', \lambda)$, we deduce that

$$M_{a''}(\lambda) = D_1 + K_1(a') V_1 \overline{K_1(a')}, \tag{15}$$

where $V_1$ is a contraction. Subtracting (14) from (15), we get

$$D_1 - D_2 = K_1(a'') U_2 \overline{K_1(a'')} - K_1(a') V_1 \overline{K_1(a')}.$$

Thus we have

$$V_1 = \left[ D_1 - D_2 + K_1(a') V_1 \overline{K_1(a')} \right].$$

Let $\Upsilon$ be a continuous mapping from the unit ball into itself such that $\Upsilon(U_2) = V_1$. Therefore, $\Upsilon$ has a unique fixed point. Letting $U_2$ and $V_1$ be replaced by $U$, we see that

$$\begin{aligned} \|D_1 - D_2\| &= \left\| K_1(a'') U \overline{K_1(a'')} - K_1(a') U \overline{K_1(a')} \right\| \\ &\le \|K_1(a'')\| \left\| \overline{K_1(a'')} - \overline{K_1(a')} \right\| \\ &\quad + \|K_1(a'') - K_1(a')\| \left\| \overline{K_1(a')} \right\|. \end{aligned}$$

As $a'$ and $a''$ approach $a$, $K_1$ and $\overline{K_1}$ have limits. The centers form a Cauchy sequence and converge.

Then, we have

$$C_2 = \pm \left[ 2 \operatorname{Im} \lambda \left( \int_0^{a'} \psi^* B \varphi d_q \zeta \right) - i I_n \right].$$

Thus at $a'$, the center

$$\begin{aligned} D = -C_3^{-1} C_2 = &- \left[ 2 \operatorname{Im} \lambda \left( \int_0^{a'} \psi^* B_1 \psi d_q \zeta \right) \right]^{-1} \\ &\times \left[ 2 \operatorname{Im} \lambda \left( \int_0^{a'} \psi^* B_1 \psi d_q \zeta \right) - i I_n \right]. \end{aligned}$$

Therefore, we obtain

$$\lim_{a' \to \infty} \mathcal{C}_{TW}(b_1', \lambda) = \mathcal{C}_{TW}^0.$$

□

It is easy to check that $M_a(\lambda) = D + K_1 U \overline{K_1}$ is well defined. As $U$ varies over the unit circle in $n \times n$ space, the limit circle or point $\mathcal{C}_{TW}^0$ is covered.

We shall now investigate the number of square-integrable solutions of the linear $q$-Hamiltonian systems.

**Theorem 12.** *Let $\chi = \varphi + \psi M$, where $M$ is a point inside $\mathcal{C}_{TW}^0 \le 0$ (see [14]). Then we have*

$$\chi \in L^2_{q,B}((0, \infty); \mathbb{C}^{2n}),$$

*where the space $L^2_{q,B}((0, \infty); \mathbb{C}^{2n})$ is defined by formula (2).*

**Proof.** Since

$$\mathcal{C}_{TW}\left(a,\lambda\right)=2\left|\operatorname{Im}\lambda\right|\left(\int_{0}^{a}\chi^{*}B\chi d_{q}\zeta\right)\pm\frac{1}{i}\left[M-M^{*}\right]\leq 0,$$

we obtain

$$0\leq\int_{0}^{a}\chi^{*}B\chi d_{q}\zeta\leq\frac{1}{2i\left|\operatorname{Im}\lambda\right|}\left[M-M^{*}\right].$$

As $a\to\infty$, the upper bound is fixed. □

**Lemma 13.** *Let* $\operatorname{rank}\overline{K_1}=r$ *and* $S\left(U\right)=K_{1}U\overline{K_{1}}$, *where* $U$ *is unitary. Then the following relations hold:*
*i)* $\operatorname{rank}S\left(U\right)\leq r$,
*ii)* $\sup\limits_{U}\operatorname{rank}S(U)=r$.

**Proof.** The proof is clear. □

**Theorem 14.** *Let* $m=n+r$. *For* $\operatorname{Im}\lambda\neq 0$, *there exist at least* $m$ *square integrable solutions of Eq.* (1), $n\leq m\leq 2n$.

**Proof.** $\varphi+D\psi$ consists of $n$ solutions in the space $L_{q,B}^{2}((0,a);\mathbb{C}^{2n})$. As $U$ varies, $\psi\left(K_{1}U\overline{K_{1}}\right)$ gives additional linearly independent $m-n$ solutions. By the reflection principles, the number of solutions is the same for $\operatorname{Im}\lambda<0$ or $\operatorname{Im}\lambda>0$. □

**5. Boundary conditions in the singular case.** Let us define $L_{\max}$ as $L_{\max}\mathcal{Z}=F$ for all $\mathcal{Z}\in\mathcal{D}_{\max}$, where

$$\mathcal{D}_{\max}:=\left\{\begin{array}{c}\mathcal{Z}\in L_{q,B}^{2}((0,\infty);\mathbb{C}^{2n}):J\mathcal{Z}^{[q]}(\zeta)-A\left(\zeta\right)\mathcal{Z}\left(\zeta\right)=B\left(\zeta\right)F\left(\zeta\right)\\ \text{exists in }(0,\infty),\ F\in L_{q,B}^{2}((0,\infty);\mathbb{C}^{2n})\end{array}\right\}.$$

**Theorem 15.** *Let* $\mathcal{Z}_j$ *be a solution of*

$$J\mathcal{Z}_{j}^{[q]}(\zeta)=\left(\overline{\lambda_{0}}B+A\right)\mathcal{Z}_{j},$$

*where* $\operatorname{Im}\lambda_{0}\neq 0$. *Then for all* $\mathcal{Z}\in\mathcal{D}_{\max}$, *the following limit*

$$A_{a_{j}}\left(\mathcal{Z}\right)=\lim_{\zeta\to\infty}\widehat{\mathcal{Z}}_{j}^{*}J\widehat{\mathcal{Z}}$$

*exists if and only if* $\mathcal{Z}_{j}\in L_{q,B}^{2}((0,\infty);\mathbb{C}^{2n})$.

**Proof.** From the following equations

$$J\mathcal{Z}^{[q]}(\zeta)-A\left(\zeta\right)\mathcal{Z}\left(\zeta\right)=B\left(\zeta\right)F\left(\zeta\right)$$

and

$$J\mathcal{Z}_{j}^{[q]}(\zeta)-A\left(\zeta\right)\mathcal{Z}_{j}\left(\zeta\right)=\overline{\lambda}B\left(\zeta\right)\mathcal{Z}_{j}\left(\zeta\right),$$

we obtain

$$\begin{aligned}
&\int_0^\zeta \mathcal{Z}_j^*(\zeta)\,B_1(\zeta)\,[F(\zeta)-\lambda\mathcal{Z}(\zeta)]\,d_q\zeta\\
&=\int_0^\zeta \mathcal{Z}_j^*(\zeta)\left[J\mathcal{Z}^{[q]}(\zeta)-A(\zeta)\,\mathcal{Z}(\zeta)\right]-\left[J\mathcal{Z}_j^{[q]}(\zeta)-A(\zeta)\,\mathcal{Z}_j(\zeta)\right]^*\mathcal{Z}(\zeta)\,d_q\zeta\\
&=\int_0^\zeta \mathcal{Z}_j^*(\zeta)\left[J\mathcal{Z}^{[q]}(\zeta)\right]d_q\zeta-\int_0^\zeta\left[J\mathcal{Z}_j^{[q]}(\zeta)\right]^*\mathcal{Z}(\zeta)\,d_q\zeta\\
&=\int_0^\zeta \left\{\mathcal{Z}_{j1}^*(\zeta)\left[-\frac{1}{q}D_{q^{-1}}\mathcal{Z}_2(\zeta)\right]+\mathcal{Z}_{j2}^*(\zeta)\,D_q\mathcal{Z}_1(\zeta)\right\}d_q\zeta\\
&\quad-\int_0^\zeta\left[\left\{-\frac{1}{q}D_{q^{-1}}\mathcal{Z}_{j2}^*(\zeta)\right\}\mathcal{Z}_1(\zeta)+D_q\mathcal{Z}_{j1}^*(\zeta)\,\mathcal{Z}_2(\zeta)\right]d_q\zeta\\
&=\int_0^\zeta\left\{\mathcal{Z}_{j1}^*(\zeta)\,[-\frac{1}{q}D_{q^{-1}}\mathcal{Z}_2(\zeta)]-D_q\mathcal{Z}_{j1}^*(\zeta)\,\mathcal{Z}_2(\zeta)\right\}d_q\zeta\\
&\quad+\int_0^\zeta\left\{\mathcal{Z}_{j2}^*(\zeta)\,D_q\mathcal{Z}_1(\zeta)-\left\{-\frac{1}{q}D_{q^{-1}}\mathcal{Z}_{j2}^*(\zeta)\right\}\mathcal{Z}_1(\zeta)\right\}d_q\zeta.
\end{aligned}$$

Since

$$\begin{aligned}
D_q\left(\mathcal{Z}_{j1}^*(\zeta)\,\mathcal{Z}_2\left(q^{-1}\zeta\right)\right)&=\left(D_q\left(q^{-1}\zeta\right)\mathcal{Z}_{j1}^*D_q\mathcal{Z}_2\left(q^{-1}\zeta\right)\right)+D_q\mathcal{Z}_{j1}^*(\zeta)\,\mathcal{Z}_2(\zeta)\\
&=\mathcal{Z}_{j1}^*(\zeta)\,[\frac{1}{q}D_{q^{-1}}\mathcal{Z}_2(\zeta)]+D_q\mathcal{Z}_{j1}^*(\zeta)\,\mathcal{Z}_2(\zeta)
\end{aligned}$$

and

$$\begin{aligned}
D_q\left(\mathcal{Z}_{j2}^*\left(q^{-1}\zeta\right)\mathcal{Z}_1(\zeta)\right)&=\left(D_q\mathcal{Z}_{j2}^*\left(q^{-1}\zeta\right)\right)D_q\left(q^{-1}\zeta\right)\mathcal{Z}_1(\zeta)\\
&\quad+\mathcal{Z}_{j2}^*(\zeta)\left(D_q\mathcal{Z}_1(\zeta)\right)\\
&=\left\{\frac{1}{q}D_{q^{-1}}\mathcal{Z}_{j2}^*(\zeta)\right\}\mathcal{Z}_1(\zeta)+\mathcal{Z}_{j2}^*(\zeta)\,D_q\mathcal{Z}_1(\zeta)\,,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^\zeta \mathcal{Z}_j^*(\zeta)\,B_1(\zeta)\,[F(\zeta)-\lambda\mathcal{Z}(\zeta)]\,d_q\zeta\\
&=\int_0^\zeta D_q\left\{\mathcal{Z}_{j2}^*\left(q^{-1}\zeta\right)\mathcal{Z}_1(\zeta)-\mathcal{Z}_{j1}^*(\zeta)\,\mathcal{Z}_2\left(q^{-1}\zeta\right)\right\}d_q\zeta\\
&=\widehat{\mathcal{Z}}_j^*J\widehat{\mathcal{Z}}(\zeta)-\widehat{\mathcal{Z}}_j^*J\widehat{\mathcal{Z}}(0)\,.
\end{aligned}\tag{16}$$

If $\mathcal{Z}_j \in L^2_{q,B}((0,\infty);\mathbb{C}^{2n})$, then as $\zeta \to \infty$, the integral in (16) converges and the limit

$$\lim_{\zeta\to\infty}(\widehat{\mathcal{Z}}_j^* J\widehat{\mathcal{Z}})(\zeta)$$

exists. Conversely, suppose that the integral in (16) converges for all $\mathcal{Z}, F \in L^2_{q,B}((0,\infty);\mathbb{C}^{2n})$. From the Hahn–Banach theorem and the Riesz representation theorem it follows that $\mathcal{Z}_j \in L^2_{q,B}((0,\infty);\mathbb{C}^{2n})$. □

Suppose that $\lambda_0$ is fixed, where $\operatorname{Im}\lambda_0 \neq 0$.

**Definition 16.** Let

$$M_a\left(\overline{\lambda}\right) = \overline{D} + \overline{K_1}UK_1$$

be on the limit circle. Let

$$\chi\left(\zeta,\overline{\lambda_0}\right) = \varphi\left(\zeta,\overline{\lambda_0}\right) + \psi\left(\zeta,\overline{\lambda_0}\right)M\left(\overline{\lambda_0}\right) \in L^2_{q,B}((0,\infty);\mathbb{C}^{2n})$$

and let $\chi\left(\zeta,\overline{\lambda_0}\right)$ satisfy the equation

$$J\mathcal{Z}^{[q]}(\zeta) = \left(\lambda_0 B(\zeta) + A(\zeta)\right)\mathcal{Z}(\zeta).$$

Then we define $S_{\lambda_0}(\mathcal{Z})$ by the formula

$$S_{\lambda_0}(\mathcal{Z}) = \lim_{\zeta\to\infty}\widehat{\chi}\left(\zeta,\overline{\lambda_0}\right)J\widehat{\mathcal{Z}}(\zeta)$$

for all $\mathcal{Z} \in \mathcal{D}_{\max}$.

**6. A self-adjoint operator.** Assume that equation (1) has $m$ solutions. Let us define $L$ as

$$\begin{aligned} L &: \mathcal{D} \to L^2_{q,B}((0,\infty);\mathbb{C}^{2n}),\\ \mathcal{Z} &\to L\mathcal{Z} = F \ \Leftrightarrow\ \Gamma(\mathcal{Z}) = BF, \end{aligned}$$

where

$$\mathcal{D} := \left\{ \begin{array}{c} \mathcal{Z} \in L^2_{q,B}((0,\infty);\mathbb{C}^{2n}) : \mathcal{Z} \text{ and } D_q\mathcal{Z} \text{ are } q\text{-regular at zero,} \\ J\mathcal{Z}^{[q]}(\zeta) - A(\zeta)\mathcal{Z}(\zeta) = B(\zeta)F(\zeta) \text{ exists in } (0,\infty), \\ F \in L^2_{q,B}((0,\infty);\mathbb{C}^{2n}),\ \Omega\widehat{\mathcal{Z}}(0) = 0, \\ \text{and } S_{\lambda_0}(\mathcal{Z}) = 0,\ \operatorname{Im}\lambda_0 \neq 0 \end{array} \right\}.$$

Thus we obtain the following theorem.

**Theorem 17.** *If* $J\mathcal{Z}^{[q]}(\zeta) - A(\zeta)\mathcal{Z}(\zeta) = B(\zeta)F(\zeta)$, $B\mathcal{Z} = 0$ *implies* $\mathcal{Z} = 0$, *then the set* $\mathcal{D}$ *is dense in* $L^2_{q,B}((0,\infty);\mathbb{C}^{2n})$.

**Proof.** Let us start with proof by contradiction. Assume that the set $\mathcal{D}$ is not dense in $L^2_{q,B}((0,\infty);\mathbb{C}^{2n})$. Then there exists a $G \in L^2_{q,B}((0,\infty);\mathbb{C}^{2n})$ such that $G$ is orthogonal to the set $\mathcal{D}$. Let $\mathcal{Y}$ satisfy $\mathcal{Y} \in \mathcal{D}$,

$$J\mathcal{Y}^{[q]}(\zeta) - A(\zeta)\mathcal{Y}(\zeta) = \overline{\lambda_0}B(\zeta)\mathcal{Y}(\zeta) + B(\zeta)G(\zeta)$$

for $\operatorname{Im}\lambda_0 \neq 0$. Then for $\mathcal{Z} \in \mathcal{D}$, we conclude that

$$\begin{aligned} 0 = (\mathcal{Z}, G) &= \int_0^\infty G^* B \mathcal{Z} d_q\zeta \\ &= \int_0^\infty \left[ J\mathcal{Y}^{[q]}(\zeta) - A(\zeta)\mathcal{Y}(\zeta) - \overline{\lambda_0} B(\zeta)\mathcal{Y}(\zeta) \right]^* \mathcal{Z} d_q\zeta \\ &= \int_0^\infty \mathcal{Y}^* \left[ J\mathcal{Z}^{[q]}(\zeta) - A(\zeta)\mathcal{Z}(\zeta) - \lambda_0 B(\zeta)\mathcal{Z}(\zeta) \right] d_q\zeta. \end{aligned}$$

Let

$$J\mathcal{Z}^{[q]}(\zeta) - A(\zeta)\mathcal{Z}(\zeta) - \lambda_0 B(\zeta)\mathcal{Z}(\zeta) = B(\zeta) F(\zeta).$$

Then we obtain

$$0 = (F, \mathcal{Y}) = \int_0^\infty \mathcal{Y}^* B F d_q\zeta. \tag{17}$$

We take $F = \mathcal{Y}$, due to the fact that $F$ is arbitrary. By (17), we see that $\mathcal{Y} = 0$ which yields $BG = 0$ and $G = 0$ in $L^2_{q,B}((0,\infty);\mathbb{C}^{2n})$. □

**Theorem 18.** *$L$ is a self-adjoint operator.*

**Proof.** Define the operator of $(L - \lambda I)^{-1}$ as

$$(L - \lambda I)^{-1} = \int_0^\infty G(\lambda, \zeta, t) B(t) F(t) d_q t, \tag{18}$$

where $\operatorname{Im}\lambda \neq 0$ and

$$G(\lambda, \zeta, t) = \begin{cases} \chi(\zeta, \lambda)\psi^*(t, \lambda), & 0 \le t \le \zeta < \infty, \\ \psi(\zeta, \lambda)\chi^*(t, \lambda), & 0 \le \zeta \le t < \infty. \end{cases}$$

Let $L\mathcal{Z} - \lambda_0\mathcal{Z} = F$ and $L^*\mathcal{Z} - \overline{\lambda_0}\mathcal{Z} = H$ ($\operatorname{Im}\lambda_0 \neq 0$). Hence

$$\begin{aligned} &\left( (L - \lambda_0 I)^{-1} F, H \right) \\ &= \int_0^\infty H^*(\zeta) B(\zeta) \left[ \int_0^\infty G(\lambda_0, \zeta, t) B(t) F(t) d_q t \right] d_q\zeta \\ &= \int_0^\infty \left[ \int_0^\infty (G(\lambda_0, \zeta, t))^* B(\zeta) H(\zeta) d_q\zeta \right]^* B(t) F(t) d_q t \\ &= \int_0^\infty \left[ \int_0^\infty (G\left(\overline{\lambda_0}, t, \zeta\right) B(\zeta) H(\zeta) d_q\zeta \right]^* B(t) F(t) d_q t \\ &= \int_0^\infty \left[ \int_0^\infty G\left(\overline{\lambda_0}, \zeta, t\right) B(t) H(t) d_q t \right]^* B(\zeta) F(\zeta) d_q\zeta \\ &= \left( F, \left(L - \overline{\lambda_0} I\right)^{-1} H \right), \end{aligned}$$

due to $G\left(\overline{\lambda_0}, t, \zeta\right) = (G(\lambda_0, \zeta, t))^*$.

Since

$$\left((L-\lambda_0 I)^{-1} F, H\right) = \left(F, \left(L^* - \overline{\lambda_0} I\right)^{-1} H\right),$$

we conclude that

$$\left(L - \overline{\lambda_0} I\right)^{-1} = \left(L^* - \overline{\lambda_0} I\right)^{-1}.$$

This implies that $L = L^*$. □

**Theorem 19.** *Let* $\operatorname{Im}\lambda_0 \neq 0$. *The operator* $(L-\lambda_0 I)^{-1}$ *defined by the formula* (18) *is a bounded operator and*

$$\left\|(L-\lambda_0 I)^{-1}\right\| \leq \frac{1}{|\operatorname{Im}\lambda_0|}.$$

**Proof.** Let $(L-\lambda_0 I)\,\mathcal{Z} = F$. Then

$$\begin{aligned}(\mathcal{Z}, F) - (F, \mathcal{Z}) &= (\mathcal{Z}, (L-\lambda_0 I)\,\mathcal{Z}) - ((L-\lambda_0 I)\,\mathcal{Z}, \mathcal{Z}) \\ &= \left(\lambda_0 - \overline{\lambda_0}\right)(\mathcal{Z}, \mathcal{Z}).\end{aligned}$$

Using Schwartz's inequality, we get

$$2\,|\operatorname{Im}\lambda_0|\,\|\mathcal{Z}\|^2 \leq 2\,\|\mathcal{Z}\|\,\|F\|.$$

Hence

$$\left\|(L-\lambda_0 I)^{-1} F\right\| \leq \frac{1}{|\operatorname{Im}\lambda_0|}\,\|F\|$$

which gives the result. □

**Theorem 20.** *Let*

$$\chi(\zeta,\lambda_0) = \varphi(\zeta,\lambda_0) + \psi(\zeta,\lambda_0)\,M(\lambda_0),$$

*where* $\operatorname{Im}\lambda_0 \neq 0$. *Then we have*

$$\lim_{\zeta\to\infty} \widehat{\chi}^*(\zeta,\lambda_0)\,J\widehat{\chi}(\zeta,\lambda_0) = 0.$$

**Proof.** Since

$$\begin{aligned}&\widehat{\chi}^*(\zeta,\lambda_0)\,J\widehat{\chi}(\zeta,\lambda_0) \\ &= \begin{pmatrix} I_n & M^*(\lambda_0) \end{pmatrix} \widehat{\mathcal{Z}}^*(\zeta,\lambda_0)\,J\widehat{\mathcal{Z}}(\zeta,\lambda_0) \begin{pmatrix} I_n \\ M(\lambda_0) \end{pmatrix} \\ &= \begin{pmatrix} I_n & M^*(\lambda_0) \end{pmatrix} J \begin{pmatrix} I_n \\ M(\lambda_0) \end{pmatrix} = 0,\end{aligned}$$

we get the desired result. □

**Availability of data and materials**

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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