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A general approach to conditional strong laws of large numbers

Abstract. A general tool to prove conditional strong laws of larger number is considered. It is shown that a conditional Kolmogorov type inequality implies a conditional Hájek–Rényi type inequality and this implies a strong law of large numbers. Both probability and moment inequalities are considered. Some applications are offered in the last section.

1. Introduction. In this paper, we study conditional strong laws of large numbers for arbitrary random variables. So, let $X_1, X_2, \dots$ be a sequence of random variables defined on the probability space $(\Omega, \mathcal{A}, P)$. The partial sums of random variables are denoted as $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$ and $S_0 = 0$.

The classic strong law of large numbers which is due to Kolmogorov asserts that, if $X_1, X_2, \dots$ are independent, identically distributed random variables with finite mean μ, then the arithmetic mean converges almost surely to μ. An elementary approach has been provided by Etemadi [1], who established the strong law of large numbers under the assumption of pairwise independence (without requiring mutual independence). For several decades, numerous findings, modifications and applications concerning the strong law of large numbers have been studied.

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In [2], Fazekas and Klesov presented a general approach to establish the strong law of large numbers for sequences of random variables. Significantly, their method does not impose any restriction on the underlying dependence structure of random variables, but it needs a Kolmogorov type inequality. Later, the approach of [2] was applied and extended by several authors, see e.g. [6]. The aim of our paper is to obtain the conditional version of the results of Fazekas and Klesov [2]. In our proofs we use the ideas given in [2].

In the last two decades several papers were devoted to conditional versions of well-known theorems of probability theory. In [3], Majerek, Nowak and Zięba studied the conditional strong law of large numbers for $\mathcal{F}$-independent random variables, where $\mathcal{F}$ is a σ-subalgebra of $\mathcal{A}$. Their main results were obtained via conditional Kolmogorov's inequality. Prakasa Rao [4] besides conditional independence, studied also conditional mixing and conditional association.

In this paper, we shall show that a conditional Kolmogorov type inequality implies a conditional Hájek–Rényi type inequality and this implies a strong law of large numbers. This approach can be used both for conditional probabilities and for conditional expectations. In the last section of this paper, we present several applications of the main result. Using our approach, we offer alternative proofs to the following theorems: the conditional strong law of large numbers for $\mathcal{F}$-independent random variables (Theorem 3.5 in [3]), a general version of the conditional strong law of large numbers (Theorem 6 in [4]) and strong law for conditionally negatively associated random variables (Theorem 3.1 (b) in [5]).

2. Conditional strong law of large numbers via the Hájek–Rényi inequality for expectations. We show that the conditional Kolmogorov inequality implies the conditional Hájek–Rényi inequality and it implies the strong law of large numbers without assuming further weak dependence conditions.

Theorem 2.1. *Let $X_1, X_2, \dots, X_n$ be random variables and $S_k = X_1 + \dots + X_k$. Let $\mathcal{F}$ be a σ-subalgebra, $\alpha_1, \dots, \alpha_n$ be nonnegative $\mathcal{F}$-measurable random variables and $r > 0$ be a real number. Assume that the general conditional Kolmogorov's type inequality is true, that is*

$$E\left(\left[\max_{1\le l\le m}|S_l|\right]^r|\mathcal{F}\right) \le \sum_{l=1}^{m}\alpha_l \qquad \textit{for all} \quad 1\le m\le n. \tag{2.1}$$

Then the conditional Hájek–Rényi inequality is true, that is

$$E\left(\left[\max_{1\le l\le n}\left|\frac{S_l}{\beta_l}\right|\right]^r|\mathcal{F}\right) \le 4\sum_{l=1}^{n}\frac{\alpha_l}{\beta_l^r} \tag{2.2}$$

for $\mathcal{F}$-measurable random variables $\beta_1 \le \beta_2 \le \ldots \le \beta_n$ with $\beta_1 \ge \beta_0$, where β_0 is a positive constant.

Proof. We can assume that $\beta_1 \geq 1$ during the proof. Let $c = 2^{\frac{1}{r}}$. Let $A_i = \{k : c^i \leq \beta_k < c^{i+1}\}$, $i = 0, 1, 2, \ldots$. Then A_i is $\mathcal{F}$-measurable, because β_k is $\mathcal{F}$-measurable. A_i is the set of subscripts k for which $c^i \leq \beta_k < c^{i+1}$. Let $i(n)$ be the index of the last nonempty A_i. Then $i(n)$ is an $\mathcal{F}$-measurable random variable (possibly having value infinity). Let $k(i)$ be the maximal index in A_i. More precisely, $k(i) = \max\{k : k \in A_i\}$, if A_i is nonempty, but $k(i) = k(i-1)$ if A_i is empty ($k(-1) = 0$ by definition). Let

$$\delta_l = \sum_{j=k(l-1)+1}^{k(l)} \alpha_j \quad \text{be the sum of } \alpha\text{'s in } A_l, \quad l = 0, 1, 2, \ldots.$$

Then $k(i)$ and δ_l are $\mathcal{F}$-measurable, $k(i) \leq n$. Using (2.1), we obtain

$$\begin{aligned}
E\left(\left[\max_{1\leq l\leq n} \frac{|S_l|}{\beta_l}\right]^r |\mathcal{F}\right) &\leq \sum_{i=0}^{i(n)} E\left(\left[\max_{l\in A_i} \frac{|S_l|}{\beta_l}\right]^r |\mathcal{F}\right) \\
&\leq \sum_{i=0}^{i(n)} c^{-ir} E\left(\left[\max_{l\in A_i} |S_l|\right]^r |\mathcal{F}\right) \leq \sum_{i=0}^{i(n)} c^{-ir} E\left(\left[\max_{k\leq k(i)} |S_k|\right]^r |\mathcal{F}\right) \\
&\leq \sum_{i=0}^{i(n)} c^{-ir} \sum_{k=1}^{k(i)} \alpha_k = \sum_{i=0}^{i(n)} c^{-ir} \sum_{l=0}^{i} \delta_l = \sum_{l=0}^{i(n)} \delta_l \sum_{i=l}^{i(n)} c^{-ir} \\
&\leq \sum_{l=0}^{i(n)} \delta_l \sum_{i=l}^{\infty} c^{-ir} = \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \delta_l \\
&= \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \sum_{k=k(l-1)+1}^{k(l)} \alpha_k \\
&\leq \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \sum_{k=k(l-1)+1}^{k(l)} \alpha_k \frac{c^{lr+r}}{\beta_k^r} \\
&= \frac{c^r}{1-c^{-r}} \sum_{l=0}^{i(n)} \sum_{k=k(l-1)+1}^{k(l)} \frac{\alpha_k}{\beta_k^r} = 4 \sum_{k=1}^{n} \frac{\alpha_k}{\beta_k^r}.
\end{aligned}$$

During the proof we applied the fact that in A_i we have $1 < \frac{c^{l+1}}{\beta_k}$. We also mention that we applied (2.1) for random number of terms, i.e., instead of m we applied it for $k(i)$. One can show that (2.1) is true for this relation, as $k(i)$ is $\mathcal{F}$-measurable and $k(i) \leq n$. $\square$

Theorem 2.2. *Let $X_1, X_2, \ldots$ be random variables, $S_n = X_1 + \cdots + X_n$ for any n. Let $b_0 \leq b_1 \leq b_2 \leq \ldots$ be $\mathcal{F}$-measurable random variables with $b_n \longrightarrow \infty$ a.s., where b_0 is a positive constant. Let $\alpha_1, \alpha_2, \ldots$ be nonnegative*

$\mathcal{F}$-measurable random variables. Let $r > 0$ be a fixed number. Assume that for any $n \geq 1$

$$E\left(\left[\max_{1\leq l\leq n}|S_l|\right]^r|\mathcal{F}\right) \leq \sum_{l=1}^{n}\alpha_l. \tag{2.3}$$

If $\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty$ a.s., then

$$\lim_{n\longrightarrow\infty}\frac{S_n}{b_n} = 0 \qquad a.s. \tag{2.4}$$

Proof. We can assume that $\alpha_n > 0$ for all n a.s. To see it take a non-random $\alpha'_n > 0$, for any n and $\sum_n \alpha'_l < \infty$. Then instead of α_n we can consider $\max\{\alpha_n, \alpha'_n\}$. Assume that $\alpha_n \geq \alpha'_n > 0$ and α'_n is non-random for any n. Let

$$v_n = \sum_{k=n}^{\infty}\frac{\alpha_k}{b_k^r}, \qquad \beta_n = \max_{1\leq k\leq n} b_k v_k^{\frac{1}{2r}}.$$

Then the sequence β_n is increasing, $\beta_1 > \beta_0 > 0$ where β_0 is non-random. Because of the assumption $\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty$ a.s., we have

$$0 < v_n < \infty \quad \text{for all } n \text{ a.s.}, \quad v_n \longrightarrow 0 \quad \text{a.s.}$$

and v_n is a decreasing sequence. Then, using the Abel–Dini theorem,

$$\sum_{n=1}^{\infty}\frac{\alpha_n}{b_n^r v_n^{\frac{1}{2}}} < \infty \quad \text{a.s.}$$

Therefore we have $0 < \beta_0 \leq \beta_1 \leq \beta_2 \leq \dots$, β_0 is non-random,

$$\sum_{k=1}^{\infty}\frac{\alpha_k}{\beta_k^r} < \infty, \quad \lim_{k\longrightarrow\infty}\frac{\beta_k}{b_k} = 0 \quad \text{a.s.}$$

Then our previous theorem implies

$$E\left(\max_{1\leq l\leq n}\left|\frac{S_l}{\beta_l}\right|^r|\mathcal{F}\right) \leq 4\sum_{l=1}^{n}\frac{\alpha_l}{\beta_l^r} \quad \text{for all } n.$$

So, by the monotone convergence theorem,

$$E\left(\sup_{1\leq l\leq\infty}\left|\frac{S_l}{\beta_l}\right|^r|\mathcal{F}\right) \leq 4\sum_{l=1}^{\infty}\frac{\alpha_l}{\beta_l^r} < \infty \quad \text{a.s.}$$

So

$$\sup_{1\leq l\leq\infty}\left|\frac{S_l}{\beta_l}\right|^r < \infty \quad \text{a.s.}$$

Therefore

$$0 \leq \left|\frac{S_l}{b_l}\right| = \left|\frac{S_l}{\beta_l}\right|\frac{\beta_l}{b_l} \leq \left(\sup_{1\leq l\leq\infty}\left|\frac{S_l}{\beta_l}\right|\right)\frac{\beta_l}{b_l} \longrightarrow 0 \quad \text{a.s. as} \quad l \longrightarrow \infty. \qquad \square$$

3. Conditional strong law of large numbers via the Hájek–Rényi inequality for probabilities. Here we offer the same approach as in the previous section, but we use conditional probabilities instead of conditional expectations.

Theorem 3.1. *Let $X_1, X_2, \dots, X_n$ be random variables, $S_k = X_1 + \dots + X_k$. Let $\mathcal{F}$ be a σ-subalgebra. Let r be a positive real number. Let $\beta_1 \le \beta_2 \le \dots \le \beta_n$ be $\mathcal{F}$-measurable, $\alpha_1, \dots, \alpha_n$ be nonnegative $\mathcal{F}$-measurable random variables. Assume that $\beta_1 \ge \beta_0 > 0$, where β_0 is non-random. If*

$$P\left(\max_{1\le l\le m} |S_l| \ge \varepsilon | \mathcal{F}\right) \le \frac{1}{\varepsilon^r} \sum_{l=1}^{m} \alpha_l \qquad \text{for all} \quad 1 \le m \le n \tag{3.1}$$

and for all $\varepsilon > 0$, then

$$P\left(\max_{1\le l\le n} \left|\frac{S_l}{\beta_l}\right| \ge \varepsilon | \mathcal{F}\right) \le \frac{4}{\varepsilon^r} \sum_{k=1}^{n} \frac{\alpha_k}{\beta_k^r} \tag{3.2}$$

for all $\varepsilon > 0$.

Proof. Using the same notation as in the proof of Theorem 2.1, we have

$$\begin{aligned}
P\left(\max_{1\le l\le n} \frac{|S_l|}{\beta_l} \ge \varepsilon | \mathcal{F}\right) &\le \sum_{i=0}^{i(n)} P\left(\max_{l\in A_i} \frac{|S_l|}{\beta_l} \ge \varepsilon | \mathcal{F}\right) \\
&\le \sum_{i=0}^{i(n)} P\left(\max_{l\in A_i} \frac{|S_l|}{c^i} \ge \varepsilon | \mathcal{F}\right) \le \sum_{i=0}^{i(n)} P\left(\max_{k\le k(i)} \frac{|S_k|}{c^i} \ge \varepsilon | \mathcal{F}\right) \\
&\le \sum_{i=0}^{i(n)} (\varepsilon c^i)^{-r} \sum_{k=1}^{k(i)} \alpha_k = \sum_{i=0}^{i(n)} (\varepsilon c^i)^{-r} \sum_{l=0}^{i} \delta_l \\
&= \sum_{l=0}^{i(n)} \delta_l \sum_{i=l}^{i(n)} (\varepsilon c^i)^{-r} \le \sum_{l=0}^{i(n)} \delta_l \sum_{i=l}^{\infty} (\varepsilon c^i)^{-r} \\
&= \varepsilon^{-r} \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \delta_l = \varepsilon^{-r} \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \sum_{k=k(l-1)+1}^{k(l)} \alpha_k \\
&\le \varepsilon^{-r} \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \sum_{k=k(l-1)+1}^{k(l)} \alpha_k \frac{c^{lr+r}}{\beta_k^r} \\
&= \varepsilon^{-r} \frac{c^r}{1-c^{-r}} \sum_{l=0}^{i(n)} \sum_{k=k(l-1)+1}^{k(l)} \frac{\alpha_k}{\beta_k^r} = 4\varepsilon^{-r} \sum_{k=1}^{n} \frac{\alpha_k}{\beta_k^r}.
\end{aligned}$$

□

Theorem 3.2. *Let $X_1, X_2, \ldots, X_n$ be random variables, $S_k = X_1 + \cdots + X_k$. Let $\mathcal{F}$ be a σ-subalgebra. Let $b_0 \le b_1 \le b_2 \ldots$ be $\mathcal{F}$-measurable random variables with $b_n \longrightarrow \infty$ a.s., where b_0 is a positive constant. Let $\alpha_1, \alpha_2, \ldots$ be nonnegative $\mathcal{F}$-measurable random variables. Let $r > 0$ be a fixed number. Assume that for any $n \ge 1$*

$$P\left(\max_{1\le l\le n} |S_l| \ge \varepsilon|\mathcal{F}\right) \le \frac{1}{\varepsilon^r}\sum_{l=1}^{n}\alpha_l \quad \textit{for all} \quad \varepsilon > 0. \tag{3.3}$$

If $\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty$ a.s., then

$$\lim_{n\longrightarrow\infty}\frac{S_n}{b_n} = 0 \quad \textit{a.s.} \tag{3.4}$$

Proof. Assume that $\alpha_n \ge \alpha_n' > 0$ where α_n' is non-random for any n. Let

$$v_n = \sum_{k=n}^{\infty}\frac{\alpha_k}{\beta_k^r}, \qquad \beta_n = \max_{1\le k\le n} b_k v_k^{\frac{1}{2r}}.$$

Then, because of the assumption $\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty$ a.s., we have

$$0 < v_n < \infty \quad \text{for all} \quad n \ge 1 \quad \text{a.s. and } v_n \longrightarrow 0 \quad \text{a.s.}$$

Moreover, the Abel–Dini's theorem implies

$$\sum_{n=1}^{\infty}\frac{\alpha_n}{b_n^r v_n^{\frac{1}{2}}} < \infty \quad \text{a.s.}$$

Therefore $\beta_1, \beta_2, \ldots$ is an increasing sequence, $\beta_1 \ge \beta_0 > 0$, where β_0 is non-random,

$$\sum_{k=1}^{\infty}\frac{\alpha_k}{\beta_k^r} < \infty, \qquad \lim_{k\longrightarrow\infty}\frac{\beta_k}{b_k} = 0 \quad \text{a.s.}$$

Then our previous theorem implies

$$P\left(\max_{1\le l\le n}\frac{|S_l|}{\beta_l} \ge \varepsilon|\mathcal{F}\right) \le \frac{4}{\varepsilon^r}\sum_{l=1}^{n}\frac{\alpha_l}{\beta_l^r} \quad \text{for all } n \text{ and } \varepsilon > 0.$$

So, by the monotone convergence theorem,

$$P\left(\sup_{1\le l<\infty}\frac{|S_l|}{\beta_l} \ge \varepsilon|\mathcal{F}\right) \le \frac{4}{\varepsilon^r}\sum_{l=1}^{\infty}\frac{\alpha_l}{\beta_l^r}.$$

Let $\varepsilon \longrightarrow \infty$, we have

$$\sup_{1\le l<\infty}\frac{|S_l|}{\beta_l} < \infty \quad \text{a.s.}$$

Now

$$0 \le \left|\frac{S_l}{b_l}\right| = \left|\frac{S_l}{\beta_l}\right|\frac{\beta_l}{b_l} \le \left(\sup_{1\le l<\infty}\frac{|S_l|}{\beta_l}\right)\frac{\beta_l}{b_l} \longrightarrow 0 \quad \text{a.s., as} \quad l \longrightarrow \infty$$

because $\frac{\beta_l}{b_l} \longrightarrow 0$ a.s. Therefore,

$$\lim_{n\longrightarrow\infty} \frac{S_n}{b_n} = 0 \quad \text{a.s.} \qquad \square$$

4. Applications. First we consider conditional Kolmogorov's strong law of large numbers for $\mathcal{F}$-independent random variables. In [3], conditionally independent random variables were studied and Kolmogorov type strong laws of large numbers were obtained. Now, we prove Theorem 3.5 of [3] using our general approach. Let $\sigma^2_{\mathcal{F}}(X) = E\{(X - E(X|\mathcal{F}))^2 \,|\mathcal{F}\}$ denote the conditional variance of X.

Theorem 4.1. *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}$-independent random variables such that $\sum_{k=1}^{\infty} \frac{\sigma^2_{\mathcal{F}}(X_k)}{k^2} < \infty$ a.s. Let $S_n = X_1 + \cdots + X_n$, $n = 1, 2, \ldots$. Then*

$$\lim_{n\longrightarrow\infty} \frac{S_n - E(S_n|\mathcal{F})}{n} = 0 \quad a.s. \tag{4.1}$$

Proof. For $\mathcal{F}$-independent random variables the Kolmogorov inequality presented in [3] is

$$P\left(\max_{1\leq k\leq n} |S_k - E(S_k|\mathcal{F})| \geq \varepsilon|\mathcal{F}\right) \leq \sum_{k=1}^{n} \frac{1}{\varepsilon^2}\sigma^2_{\mathcal{F}}(X_k). \tag{4.2}$$

Then (4.2) is the condition (3.1) for $r = 2$. As $\sum_{k=1}^{\infty} \frac{\sigma^2_{\mathcal{F}}(X_k)}{k^2} < \infty$ a.s., we can apply Theorem 3.2. Therefore

$$\lim_{n\longrightarrow\infty} \frac{S_n - E(S_n|\mathcal{F})}{n} = 0 \quad \text{a.s.} \qquad \square$$

Remark 4.2. By using Theorem 3.1, we can obtain the Hájek–Rényi type inequality for conditionally independent random variables as

$$P\left(\max_{1\leq k\leq n} \left|\frac{S_k - E(S_k|\mathcal{F})}{k}\right| \geq \varepsilon|\mathcal{F}\right) \leq \frac{4}{\varepsilon^2} \sum_{k=1}^{n} \frac{\sigma^2_{\mathcal{F}}(X_k)}{k^2}.$$

Prakasa Rao in [4] obtained a general version of the conditional strong law of large numbers proved in [3]. We apply our Theorem 3.2 to prove the following theorem (Theorem 6 in [4]).

Theorem 4.3. *If $\{X_n, n \geq 1\}$ is a sequence of $\mathcal{F}$-independent random variables such that*

$$\sum_{n=1}^{\infty} \frac{E\left(|X_n - E(X_n|\mathcal{F})|^{2r} \,|\mathcal{F}\right)}{n^{r+1}} < \infty \quad a.s., \tag{4.3}$$

for some $r \geq 1$, then

$$\frac{S_n - E(S_n|\mathcal{F})}{n} \longrightarrow 0 \quad a.s. \;\; as \quad n \longrightarrow \infty. \tag{4.4}$$

Proof. By the Kolmogorov inequality in Theorem 4 of [4] for $r \geq 1$, and by inequality (5.1) of [4]

$$P\left(\max_{1\leq k\leq n} |S_k - E(S_k|\mathcal{F})| \geq \varepsilon|\mathcal{F}\right) \leq \frac{1}{\varepsilon^{2r}} E\left(|S_n - E(S_n|\mathcal{F})|^r |\mathcal{F}\right)$$
$$\leq \frac{1}{\varepsilon^{2r}} n^{r-1} \sum_{k=1}^{n} E\left(|X_k - E(X_k|\mathcal{F})|^{2r} |\mathcal{F}\right) = \frac{1}{\varepsilon^{2r}} \Lambda_n.$$

We want to represent Λ_n as $\Lambda_n = \alpha_1 + \cdots + \alpha_n$. Let

$$A_k = E\left(|X_k - E(X_k|\mathcal{F})|^{2r} |\mathcal{F}\right),$$

then

$$\alpha_n = \Lambda_n - \Lambda_{n-1} = n^{r-1} \sum_{k=1}^{n} A_k - (n-1)^{r-1} \sum_{k=1}^{n-1} A_k$$
$$= n^{r-1} A_n + [n^{r-1} - (n-1)^{r-1}] \sum_{k=1}^{n-1} A_k.$$

We have to show, that $\sum_{n=1}^{\infty} \frac{\alpha_n}{n^{2r}} < \infty$.

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{n^{2r}} = \sum_{n=1}^{\infty} \frac{n^{r-1}}{n^{2r}} A_n + \sum_{n=1}^{\infty} \frac{n^{r-1} - (n-1)^{r-1}}{n^{2r}} \sum_{k=1}^{n-1} A_k.$$

Changing the order of the summation in the second term, we obtain

$$\sum_{k=1}^{\infty} A_k \sum_{n=k+1}^{\infty} \frac{n^{r-1} - (n-1)^{r-1}}{n^{2r}} \leq \sum_{k=1}^{\infty} A_k \sum_{n=k+1}^{\infty} \frac{Cn^{r-2}}{n^{2r}}$$
$$= C \sum_{k=1}^{\infty} A_k \sum_{n=k+1}^{\infty} n^{-r-2}$$
$$\leq C \sum_{k=1}^{\infty} A_k \int_k^{\infty} x^{-r-2} dx$$
$$\leq C \sum_{k=1}^{\infty} A_k k^{-r-1},$$

where we used the mean value theorem and approximation with integral. So

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{n^{2r}} \leq C \sum_{n=1}^{\infty} \frac{A_n}{n^{r+1}} < \infty$$

using condition (4.3). Hence

$$P\left(\max_{1\leq k\leq n} |S_k - E(S_k|\mathcal{F})| \geq \varepsilon|\mathcal{F}\right) \leq \frac{1}{\varepsilon^{2r}} C \sum_{k=1}^{n} \alpha_k,$$

where $\sum_{n=1}^{\infty} \frac{\alpha_n}{n^{2r}} < \infty$. So our Theorem 3.2 implies

$$\frac{S_n - E(S_n|\mathcal{F})}{n} \longrightarrow 0 \quad \text{a.s. as} \quad n \longrightarrow \infty. \qquad \square$$

Now we show that our approach gives a quick proof of Theorem 3.1 (b) of [5].

Theorem 4.4. *Let b_n be an increasing sequence of positive real numbers, $b_n \longrightarrow \infty$. Let $X_1, X_2, \dots$ be conditionally centered $\mathcal{F}$-negatively associated random variables, $1 \le r \le 2$. Assume that $\sum_{n=1}^{\infty} \frac{E(|X_n|^r|\mathcal{F})}{b_n^r} < \infty$ a.s. Then*

$$\frac{1}{b_n} \sum_{k=1}^{n} X_k \longrightarrow 0 \quad \textit{a.s. as} \quad n \longrightarrow \infty. \tag{4.5}$$

Proof. For our random variables the following Kolmogorov-type inequality is true

$$E\left(\max_{1\le k\le n} \left| \sum_{i=1}^{k} X_i \right|^r |\mathcal{F} \right) \le C \sum_{i=1}^{k} E\left(|X_i|^r \,|\mathcal{F}\right) \quad \text{a.s.},$$

see Lemma 2.1 of [5]. Then our Theorem 2.2 gives the result without any further calculation. $\square$

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