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The Legendre maps from two Lagrangians or from a Lagrangian and a p-form

Abstract. Let $\mathcal{FM}_{m,n}$ denote the category of fibered manifolds with m-dimensional bases and n-dimensional fibres and their fibered local diffeomorphisms. We prove that if m, n and s are positive integers, then any $\mathcal{FM}_{m,n}$-natural operator C transforming tuples (λ_1, λ_2) of Lagrangians $\lambda_1, \lambda_2 : J^sY \to \bigwedge^m T^*M$ on $\mathcal{FM}_{m,n}$-objects $Y \to M$ into Legendre maps $C(\lambda_1, \lambda_2) : J^sY \to S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M$ on Y is of the form $C(\lambda_1, \lambda_2) = c_1\Lambda(\lambda_1) + c_2\Lambda(\lambda_2)$, $c_1, c_2 \in \mathbf{R}$, where Λ is the Legendre operator. We also prove that if m, n, s and p are positive integers, then any $\mathcal{FM}_{m,n}$-natural operator C transforming tuples (λ, η) of Lagrangians $\lambda : J^sY \to \bigwedge^m T^*M$ and p-forms $\eta \in \Omega^p(M)$ into Legendre maps $C(\lambda, \eta) : J^sY \to S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M$ is of the form $C(\lambda, \eta) = c\Lambda(\lambda)$, $c \in \mathbf{R}$, where Λ is the Legendre operator.

1. Introduction. All manifolds considered in this paper are assumed to be finite dimensional and smooth (i.e. of class $\mathcal{C}^\infty$). Mappings between manifolds are assumed to be smooth (of class $\mathcal{C}^\infty$).

For a fibred manifold $Y \to M$, we have the s-jet prolongation J^sY of $Y \to M$ (for a positive integer s) and the vertical bundle $VY \to Y$ and its dual bundle $V^*Y \to Y$ and the tangent bundle TM and its symmetric sth product S^sTM and the cotangent bundle T^*M and its mth inner product $\bigwedge^m T^*M$. Given fibred manifolds $Z_1 \to M$ and $Z_2 \to M$, let $\mathcal{C}^\infty_M(Z_1, Z_2)$

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denote the space of all base preserving fibred maps of Z_1 into Z_2. Let $m = \dim(M)$. Elements from $\mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M)$ are called (sth order) Lagrangians on $Y \to M$ and elements from $\mathcal{C}^\infty_Y(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M)$ are called sth order Legendre maps on $Y \to M$.

Let $\mathcal{FM}_{m,n}$ be the category of fibred manifolds with m-dimensional bases and n-dimensional fibres and their fibred local diffeomorphisms. Any sth order Lagrangian $\lambda : J^sY \to \bigwedge^m T^*M$ on an $\mathcal{FM}_{m,n}$-object $Y \to M$ induces canonically the Legendre map $\Lambda(\lambda) : J^sY \to S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M$. Then we have the so-called Legendre operator

$$\Lambda : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right).$$

In [6], it is proved that if m, n and s are positive integers, then any regular local $\mathcal{FM}_{m,n}$-natural operator

$$A : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right)$$

is of the form $c\Lambda$, $c \in \mathbf{R}$, where Λ is the Legendre operator.

In [1], we deduced that if m, n and s are positive integers with $m \geq 3$, then any local $\mathcal{FM}_{m,n}$-natural regular operator

$$B : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \times \mathcal{C}^\infty(M, \mathbf{R}) \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right)$$

(transforming an sth order Lagrangian λ on a $\mathcal{FM}_{m,n}$-object $Y \to M$ and a map $g : M \to M$ into an sth order Legendre map $B(\lambda, g)$ on $Y \to M$) is of the form

$$B(\lambda, g)_{|j^s_{x_o}\sigma} = h(g(x_o)) \cdot \Lambda(\lambda)_{|j^s_{x_o}\sigma}$$

for a map $h : \mathbf{R} \to \mathbf{R}$, where Λ is the Legendre operator.

In the present paper, we study how a tuple (λ_1, λ_2) of Lagrangians λ_1 and λ_2 on a $\mathcal{FM}_{m,n}$-object $Y \to M$ can induce canonically a Legendre map $C(\lambda_1, \lambda_2)$ on $Y \to M$. The main result is the following theorem.

Theorem 1.1. *Let m, n, s be positive integers. Any regular local $\mathcal{FM}_{m,n}$-natural operator*

$$C : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \times \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right)$$

is of the form $C(\lambda_1, \lambda_2) = c_1\Lambda(\lambda_1) + c_2\Lambda(\lambda_2)$, $c_1, c_2 \in \mathbf{R}$, where Λ is the Legendre operator.

The proof of the above theorem will be given in Section 3.

We recall that the s-jet prolongation J^sY of a fibred manifold $Y \to M$ is the space of all s-jets $j^s_x\sigma$ at $x \in M$ of local sections $\sigma : M \to Y$ of $Y \to M$. There is the source projection $J^sY \to M$ defined by $j^s_x\sigma \mapsto x$. Consequently, J^sY is the fibre manifold with the base M. Let $Y \to M$ and $Y^1 \to M_1$ be fibred manifolds with m-dimensional bases M and M_1. Any fibred map $f : Y \to Y^1$ with the base map $\underline{f} : M \to M_1$ being local diffeomorphism induces $J^sf : J^sY \to J^sY_1$ given by $J^rf(j^s_{x_o}\sigma) = j^s_{\underline{f}(x_o)}(f \circ \sigma \circ \underline{f}^{-1})$, $j^s_{x_o}\sigma \in J^s_{x_o}Y$, $x_o \in M$.

The concept of natural operators can be found in [3]. In our case, the $\mathcal{FM}_{m,n}$-naturality (invariance) of C means that for any $\mathcal{FM}_{m,n}$-map $f : Y \to Y_1$ with the base map $\underline{f} : M \to M_1$ and Lagrangians $\lambda_1, \lambda_2 \in \mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M)$ and $\lambda'_1, \lambda'_2 \in \mathcal{C}^\infty_{M_1}(J^sY_1, \bigwedge^m T^*M_1)$, if λ_1 and λ'_1 are f-related (i.e. $\bigwedge^m T^*\underline{f} \circ \lambda_1 = \lambda'_1 \circ J^sf$) and λ_2 and λ'_2 are f-related, then so are $C(\lambda_1, \lambda_2)$ and $C(\lambda'_1, \lambda'_2)$ (i.e. $(S^sT\underline{f} \otimes V^*f \otimes \bigwedge^m T^*\underline{f}) \circ C(\lambda_1, \lambda_2) = C(\lambda'_1, \lambda'_2) \circ J^sf$). The locality of C means that $C(\lambda_1, \lambda_2)_\rho$ depends on $\mathrm{germ}_\rho(\lambda_1, \lambda_2)$ for any $\rho \in J^sY$ and $\lambda_1, \lambda_2 \in \mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M)$. The regularity means that C transforms smoothly parametrized families of tuples of Lagrangians into smoothly parametrized families of Legendre maps.

The Legendre map $\Lambda(\lambda) : J^sY \to S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M$ of a Lagrangian $\lambda : J^sY \to \bigwedge^m T^*M$ on a $\mathcal{FM}_{m,n}$-object $Y \to M$ can be constructed as follows (see, e.g. [1]). Let $\delta\lambda : \mathcal{C}^\infty_{J^sY}(J^sY, V^*J^sY \otimes \bigwedge^m T^*M)$ denote the vertical differential of λ, i.e. the composition of the restriction $\tilde{\delta}\lambda : VJ^sY \to V\bigwedge^m T^*M = \bigwedge^m T^*M \times_M \bigwedge^m T^*M$ of the differential $d\lambda : TJ^sY \to T\bigwedge^m T^*M$ of λ to the vertical sub-bundles with the second (essential) factor projection $\bigwedge^m T^*M \times_M \bigwedge^m T^*M \to \bigwedge^m T^*M$. Then $\Lambda(\lambda) : S^sT^*M \otimes VY \to \bigwedge^m T^*M$ is defined to be the restriction of $\delta\lambda : VJ^sY \to \bigwedge^m T^*M$ to the vector-subbundle $S^sT^*M \otimes VY \subset VJ^sY$, the kernel of $V\pi^s_{s-1} : VJ^sY \to VJ^{s-1}Y$, where $\pi^s_{s-1} : J^sY \to J^{s-1}Y$ is the jet projection.

The Legendre map (transformation) $\Lambda(\lambda)$ plays an important role in analytical mechanics, especially in the case of regular Lagrangians λ the transformation $\Lambda(\lambda)$ can be considered as the corresponding $J^{s-1}Y$-preserving diffeomorphism between J^sY and $(\pi^{s-1}_0)^*(S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M)$ (then it joints the Lagrange and Hamilton formalisms in fibred manifolds), see [2].

In Section 4, modifying respectively the proof of Theorem 1.1, we also prove the following p-form version of the mentioned above result of [1].

Theorem 1.2. *Let m, n, s, p be positive integers. Any regular local $\mathcal{FM}_{m,n}$-natural operator*

$$C : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \times \Omega^p(M) \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right)$$

(transforming tuples (λ, η) of Lagrangians λ on $\mathcal{FM}_{m,n}$-objects $Y \to M$ and p-forms η on M into Legendre maps $C(\lambda, \eta)$ on $Y \to M$) is of the form $C(\lambda, \eta) = c\Lambda(\lambda)$, $c \in \mathbf{R}$, where Λ is the Legendre operator.

2. Preparation. From now on, let $\mathbf{N} = \{0, 1, 2, \dots\}$ and let $\mathbf{R}^{m,n}$ be the trivial (affine) bundle $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ and let $x^1, \dots, x^m, y^1, \dots, y^n$ be the usual coordinates on $\mathbf{R}^{m,n}$. Let $dx^\mu = dx^1 \wedge \dots \wedge dx^m$. Given $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{N}^m$, let $x^\alpha := (x^1)^{\alpha_1} \cdot \dots \cdot (x^m)^{\alpha_m}$. Given $i = 1, \dots, m$ let $1_i := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbf{N}^m$, where 1 occupies ith position.

On $J^s(\mathbf{R}^{m,n})$ we have the induced coordinates $((x^i), (y^j_\alpha))$, where $i = 1, \dots, m$ and $j = 1, \dots, n$ and $\alpha = (\alpha_1, \dots \alpha_m) \in \mathbf{N}^m$ are such that $|\alpha| = \alpha_1 + \dots + \alpha_m \le s$. They are defined by

$$x^i(j^s_{x_o}\sigma) := x^i_o \quad \text{and} \quad y^j_\alpha(j^s_{x_o}\sigma) := (\partial_\alpha \sigma^j)(x_o)$$

for any $j^s_{x_o}\sigma = j^s_{x_o}(\sigma^1, \dots, \sigma^n) \in J^s_{x_o}(\mathbf{R}^{m,n}) = J^s_{x_o}(\mathbf{R}^m, \mathbf{R}^n)$, $x_o \in \mathbf{R}^m$, where ∂_α is the iterated partial derivative as indicated multiplied by $\frac{1}{\alpha!}$.

Lemma 2.1 ([4]). *Let $i = 1, \dots, m$ and $j = 1, \dots, n$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{N}^m$ be such that $|\alpha| \le s$.*

(i) For any $\tau = (\tau^1, \dots, \tau^n) \in (\mathbf{R} \setminus \{0\})^n$, we have

$$(J^s\psi_\tau)_* y^j_\alpha = \tau^j y^j_\alpha \ ,$$

where $\psi_\tau = (x^1, \dots, x^m, \frac{1}{\tau^1} y^1, \dots, \frac{1}{\tau^n} y^n)$ is the $\mathcal{FM}_{m,n}$-map.

(ii) For any $t \in \mathbf{R} \setminus \{0\}$, we have

$$(J^s\varphi^i_t)_* y^j_\alpha = t^{-\alpha_i} y^j_\alpha \ ,$$

where $\varphi^i_t = (x^1, \dots, \frac{1}{t} x^i, \dots, x^m, y^1, \dots, y^n)$ is the $\mathcal{FM}_{m,n}$-map.

3. Proof of Theorem 1.1.

Proof. Using the invariance of C with respect to the $\mathcal{FM}_{m,n}$-charts, we conclude that C is determined by the collection of values

$$\langle C(\lambda_1, \lambda_2)_\rho, \otimes^s d_0\omega \otimes v \rangle \in \bigwedge^m T^*_0\mathbf{R}^m$$

for all $\lambda_1, \lambda_2 \in \mathcal{C}^\infty_{\mathbf{R}^m}(J^s(\mathbf{R}^{m,n}), \bigwedge^m T^*\mathbf{R}^m)$ and $d_0\omega \in T^*_0\mathbf{R}^m$ and $v \in T_0\mathbf{R}^n = V_{(0,0)}\mathbf{R}^{m,n}$ and $\rho = j^s_0(\sigma) \in J^s_0(\mathbf{R}^m, \mathbf{R}^n)_0 = J^s_{(0,0)}(\mathbf{R}^{m,n})$. (The phrase "$C$ is determined by..." means that if C' is an another operator in question giving the same as C collection of values, then $C = C'$.)

Given $\rho = j^s_0(\sigma) \in J^s_0(\mathbf{R}^m, \mathbf{R}^n)_0 = J^s_{(0,0)}(\mathbf{R}^{m,n})$, there is an $\mathcal{FM}_{m,n}$-map $\nu : \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ sending $j^s_0(\sigma)$ to $\theta := j^s_0(0) \in J^s_0(\mathbf{R}^m, \mathbf{R}^n)_0 = J^s_{(0,0)}(\mathbf{R}^{m,n})$. (Indeed, we can put $\nu := (x, y - \sigma(x))$, where $x = (x^1, \dots, x^m)$ and $y = (y^1, \dots, y^n)$.) Then we can additionally assume $\rho = \theta := j^s_0(0)$.

One can additionally assume $v \ne 0$. Then using the invariance of C with respect to an $\mathcal{FM}_{m,n}$-map Φ of the form $\mathrm{id}_{\mathbf{R}^m} \times \phi$ with a respective linear isomorphism $\phi : \mathbf{R}^n \to \mathbf{R}^n$, one can additionally assume $v = \frac{\partial}{\partial y^1}_{|(0,0)}$ (because Φ preserves θ).

Similarly, one can additionally assume $d_0\omega \neq 0$. Then using the invariance of C with respect to a $\mathcal{FM}_{m,n}$-map Φ of the form $\chi \times \mathrm{id}_{\mathbf{R}^n}$ with a respective linear isomorphism $\chi : \mathbf{R}^m \to \mathbf{R}^m$, one can additionally assume $d_0\omega = d_0x^m$ (because Φ preserves θ and $v = \frac{\partial}{\partial y^1}_{|(0,0)}$).

One can write $\lambda_1 = L_1((x^i), (y^j_\alpha))dx^\mu + f_1(x^1, \dots, x^m)dx^\mu$, where f_1 is an arbitrary real valued map and L_1 is an arbitrary admissible map, i.e. such that $L_1((x^i), (0)) = 0$. Similarly, one can write $\lambda_2 = L_2((x^i), (y^j_\alpha))dx^\mu + f_2(x^1, \dots, x^m)dx^\mu$, where f_2 is an arbitrary real valued map and L_2 is an arbitrary admissible map, i.e. such that $L_2((x^i), (0)) = 0$.

Because of the locality of C, applying the result of [5], one can additionally assume that L_1 and L_2 and f_1 and f_2 are arbitrary polynomials in $((x^i), (y^j_\alpha))$ and in (x^i) (respectively) of degree $\leq q$, where q is an arbitrary positive integer.

Using the invariance of C with respect to $\psi_\tau = (x^1, \dots, x^m, \frac{1}{\tau^1}y^1, \dots, \frac{1}{\tau^n}y^n)$ being $\mathcal{FM}_{m,n}$-map for any $(\tau^1, \dots, \tau^n) \in (\mathbf{R} \setminus \{0\})^n$, one can obtain the homogeneity condition

$$\begin{aligned}
&\Big\langle C(L_1((x^i), (\tau^j y^j_\alpha))dx^\mu + f_1(x^1, \dots, x^m)dx^\mu, L_2((x^i), (\tau^j y^j_\alpha))dx^\mu \\
&\qquad + f_2(x^1, \dots, x^m)dx^\mu,)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \Big\rangle \\
&= \tau^1 \Big\langle C(L_1((x^i), (y^j_\alpha))dx^\mu + f_1(x^1, \dots, x^m)dx^\mu, L_2((x^i), (y^j_\alpha))dx^\mu \\
&\qquad + f_2(x^1, \dots, x^m)dx^\mu,)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \Big\rangle,
\end{aligned}$$

see Lemma 2.1 (i). Then, applying the homogeneous function theorem (see [3]), we conclude that

$$\Big\langle C(L_1dx^\mu + f_1dx^\mu, L_2dx^\mu + f_2dx^\mu)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \Big\rangle$$

depends linearly on admissible (L_1, L_2) (i.e. for any (f_1, f_2) the map

$$(L_1, L_2) \to \Big\langle C(L_1dx^\mu + f_1dx^\mu, L_2dx^\mu + f_2dx^\mu)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \Big\rangle,$$

where L_1, L_2 are admissible, is linear), and that C is determined by the collection of values

$$\Big\langle C(x^\beta y^1_\alpha dx^\mu + f_1(x^1, \dots, x^m)dx^\mu, f_2(x^1, \dots, x^m)dx^\mu)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \Big\rangle$$

and

$$\left\langle C(f_1(x^1,\dots,x^m)dx^\mu, x^\beta y^1_\alpha dx^\mu + f_2(x^1,\dots,x^m)dx^\mu)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle$$

for all $\alpha,\beta \in \mathbf{N}^m$ with $|\beta|\leq q$ and $|\alpha|\leq s$ and all f_1 and f_2 as above.

We can see that $\varphi^i_t := (x^1,\dots,\frac{1}{t}x^i,\dots,x^m,y^1,\dots,y^n)$ preserves C and θ and $\frac{\partial}{\partial y^1}_{|(0,0)}$ and it sends x^β into $t^{\beta_i}x^\beta$ and it sends x^i into tx^i and it preserves $x^1,\dots,x^{i-1},x^{i+1},\dots,x^m$ and it sends y^1_α into $t^{-\alpha_i}y^1_\alpha$ and it sends dx^μ into tdx^μ and it sends $\otimes^s d_0x^m$ into $t^{\delta_{im}s}\otimes^s d_0x^m$ (the Kronecker delta), see Lemma 2.1 (ii). So, using the invariance of C with respect to φ^i_t and the fact that $\langle C(L_1dx^\mu + f_1dx^\mu, f_2dx^\mu)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)}\rangle$ depends linearly on admissible (L_1,L_2), we get the condition

$$\begin{aligned} t^{\kappa_i}\Big\langle C(x^\beta y^1_\alpha dx^\mu + tf_1(x^1,\dots,tx^i,\dots,x^m)dx^\mu,\\ tf_2(\dots,tx^i,\dots)dx^\mu)_\theta, \otimes^s d_0x^m\otimes\frac{\partial}{\partial y^1}_{|(0,0)}\Big\rangle\\ = \Big\langle C(x^\beta y^1_\alpha dx^\mu + f_1(x^1,\dots,x^m)dx^\mu,\\ f_2(x^1,\dots,x^m)dx^\mu)_\theta, \otimes^s d_0x^m\otimes\frac{\partial}{\partial y^1}_{|(0,0)}\Big\rangle, \end{aligned}$$

where

$$\kappa_i = \beta_i - \alpha_i + \delta_{im}s\,.$$

Then putting $t\to 0$, we obtain

$$\begin{aligned} \Big\langle C(x^\beta y^1_\alpha dx^\mu + f_1(x^1,\dots,x^m)dx^\mu,\\ f_2(x^1,\dots,x^m)dx^\mu)_\theta, \otimes^s d_0x^m\otimes\frac{\partial}{\partial y^1}_{|(0,0)}\Big\rangle = 0 \end{aligned}$$

for any $\beta,\alpha\in\mathbf{N}^m$ with both $|\alpha|\leq s$ and $\kappa_i>0$ for some $i=1,\dots,m$. Moreover,

$$\begin{aligned} \Big\langle C(x^\beta y^1_\alpha dx^\mu + f_1(x^1,\dots,x^m)dx^\mu,\\ f_2(x^1,\dots,x^m)dx^\mu)_\theta, \otimes^s d_0x^m\otimes\frac{\partial}{\partial y^1}_{|(0,0)}\Big\rangle\\ = \Big\langle C(x^\beta y^1_\alpha dx^\mu, 0dx^\mu)_\theta, \otimes^s d_0x^m\otimes\frac{\partial}{\partial y^1}_{|(0,0)}\Big\rangle \end{aligned}$$

if additionally $\kappa_i = 0$ for some $i=1,\dots,m$.

Similarly, we obtain

$$\left\langle C(f_1(x^1,\dots,x^m)dx^\mu, x^\beta y^1_\alpha dx^\mu + f_2(x^1,\dots,x^m)dx^\mu)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle = 0$$

for any $\beta,\alpha \in \mathbf{N}^m$ with both $|\alpha| \le s$ and $\kappa_i > 0$ for some $i = 1,\dots,m$. Moreover,

$$\left\langle C(f_1(x^1,\dots,x^m)dx^\mu, x^\beta y^1_\alpha dx^\mu + f_2(x^1,\dots,x^m)dx^\mu)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$
$$= \left\langle C(0dx^\mu, x^\beta y^1_\alpha dx^\mu)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$

if additionally $\kappa_i = 0$ for some $i = 1,\dots,m$.

Consequently, C is determined by the collection of values

$$\left\langle C(x^\beta y^1_\alpha dx^\mu + f_1(x^1,\dots,x^m)dx^\mu, f_2(x^1,\dots,x^m)dx^\mu)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$

and

$$\left\langle C(f_1(x^1,\dots,x^m)dx^\mu, x^\beta y^1_\alpha dx^\mu + f_2(x^1,\dots,x^m)dx^\mu)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$

for all f_1, f_2 as above and for all $\alpha,\beta \in \mathbf{N}^m$ with $|\alpha| \le s$ and $\kappa_1 \le 0, \dots, \kappa_m \le 0$, i.e. for all f_1 and f_2 as above and for $\beta = (0,\dots,0)$ and $\alpha = (0,\dots,0,s)$.

Consequently, C is determined by the collection of values

$$\left\langle C(y^1_{(0,\dots,0,s)} dx^\mu, 0dx^\mu)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle \in \bigwedge^m T^*_0\mathbf{R}^m$$

and

$$\left\langle C(0dx^\mu, y^1_{(0,\dots,0,s)} dx^\mu)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle \in \bigwedge^m T^*_0\mathbf{R}^m\,.$$

Consequently, the vector space of all C in question is of dimension ≤ 2. So, the dimension argument ends the proof of our theorem. □

4. Proof of Theorem 1.2.

Schema of the proof. We will proceed quite similarly as in Section 3.

Using the invariance of C with respect to the $\mathcal{FM}_{m,n}$-charts, C is determined by the collection of values

$$\left\langle C(\lambda,\eta)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle \in \bigwedge^m T^*_0\mathbf{R}^m$$

for all $\lambda \in \mathcal{C}^\infty_{\mathbf{R}^m}(J^s(\mathbf{R}^{m,n}), \bigwedge^m T^*\mathbf{R}^m)$ and $\eta \in \Omega^p(\mathbf{R}^m)$, where $\theta := j^s_0(0)$.

One can write $\lambda = L((x^i), (y^j_\alpha))dx^\mu + f(x^1, \dots, x^m)dx^\mu$, where f is an arbitrary real valued map and L is an arbitrary admissible map, i.e. such that $L((x^i), (0)) = 0$. Because of the locality of C, applying the result of [5], one can additionally assume that L and f and the coefficients of η are arbitrary polynomials in $((x^i), (y^j_\alpha))$ and in (x^i) (respectively) of degree $\le q$, where q is an arbitrary positive integer.

Using the invariance of C with respect to $\psi_\tau = (x^1, \dots, x^m, \frac{1}{\tau^1}y^1, \dots, \frac{1}{\tau^n}y^n)$ being $\mathcal{FM}_{m,n}$-map for any $(\tau^1, \dots, \tau^n) \in (\mathbf{R}\backslash\{0\})^n$, we get the homogeneity condition

$$\left\langle C(L((x^i), (\tau^j y^j_\alpha))dx^\mu + f(x^1, \dots, x^m)dx^\mu, \eta)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$
$$= \tau^1 \left\langle C(L((x^i), (y^j_\alpha))dx^\mu + f(x^1, \dots, x^m)dx^\mu, \eta)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle,$$

see Lemma 2.1 (i). Then, applying the homogeneous function theorem (see [3]), we conclude that $\langle C(Ldx^\mu + f(x^1, \dots, x^m)dx^\mu, \eta)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\rangle$ depends linearly on admissible L, and that C is determined by the collection of values

$$\left\langle C(x^\beta y^1_\alpha dx^\mu + f(x^1, \dots, x^m)dx^\mu, \eta)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$

for all $\alpha, \beta \in \mathbf{N}^m$ with $|\beta| \le q$ and $|\alpha| \le s$ and all f and η as above.

Then, by the invariance of C with respect to $\varphi^i_t := (x^1, \dots, \frac{1}{t}x^i, \dots, x^m, y^1, \dots, y^n)$, we get the condition

$$t^{\kappa_i} \left\langle C(x^\beta y^1_\alpha dx^\mu + tf(x^1, \dots, tx^i, \dots, x^m)dx^\mu, (\varphi^i_t)_*\eta)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$
$$= \left\langle C(x^\beta y^1_\alpha dx^\mu + f(x^1, \dots, x^m)dx^\mu, \eta)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle,$$

where $\kappa_i := \beta_i - \alpha_i + \delta_{im}s$, $t > 0$, $i = 1, \dots, m$.

Then putting $t \to 0$, we obtain

$$\left\langle C(x^\beta y^1_\alpha dx^\mu + f(x^1, \dots, x^m)dx^\mu, \eta)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle = 0$$

for any $\beta, \alpha \in \mathbf{N}^m$ with both $|\alpha| \le s$ and $\kappa_i > 0$ for some $i = 1, \dots, m$. Moreover,

$$\left\langle C(x^\beta y^1_\alpha dx^\mu + f(x^1, \dots, x^m)dx^\mu, \eta)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$
$$= \left\langle C(x^\beta y^1_\alpha dx^\mu, (x^1, \dots, 0x^i, \dots, x^m)^*\eta)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$

if $\kappa_i = 0$, where $i = 1, \dots, m$.

Consequently, C is determined by the value

$$\left\langle C(y^1_{(0,\dots,0,s)}dx^\mu, 0)_\theta, \otimes^s d_0x^m \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle \in \bigwedge^m T^*_0\mathbf{R}^m \ .$$

Consequently, the vector space of all C in question is of dimension ≤ 1. So, the dimension argument ends the proof of our theorem. □

5. Generalizations. Theorem 1.1 can be generalized to the following:

Theorem 5.1. *Let m, n, s be positive integers. Any regular local $\mathcal{FM}_{m,n}$-natural operator*

$$D : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \times \dots \times \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \\ \to \mathcal{C}^\infty_Y\left(J^{2s}Y, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right)$$

is of the form $D(\lambda_1, \dots, \lambda_k) = c_1\Lambda(\lambda_1) + \dots + c_k\Lambda(\lambda_k)$, $c_1, \dots, c_k \in \mathbf{R}$, where Λ is the Legendre operator.

Proof. The proof of Theorem 5.1 is an obvious modification of the one of Theorem 1.1 and it is left to the reader. □

Theorem 1.2 can be generalized to the following:

Theorem 5.2. *Let $m, n, s, p_1, \dots, p_k$ be positive integers. Any regular local $\mathcal{FM}_{m,n}$-natural operator*

$$C : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \times \Omega^{p_1}(M) \times \dots \times \Omega^{p_k}(M) \\ \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right)$$

is of the form $C(\lambda, \eta_1, \dots, \eta_k) = c\Lambda(\lambda)$, $c \in \mathbf{R}$, where Λ is the Legendre operator.

Clearly, Theorem 5.2 is an immediate consequence of the following more general:

Theorem 5.3. *Let m, n, s be positive integers and let $F : \mathcal{M}f \to \mathcal{VB}$ be a vector bundle functor with the point property. Any regular local $\mathcal{FM}_{m,n}$-natural operator*

$$C : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \times \mathcal{C}^\infty_M((FM)^*) \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right)$$

where $\mathcal{C}^\infty_M((FM)^*)$ is the space of all smooth section of the vector bundle $(FM)^* \to M$ (dual to $FM \to M$), is of the form $C(\lambda, \eta) = c\Lambda(\lambda)$, $c \in \mathbf{R}$, where Λ is the Legendre operator.

Proof. The proof of Theorem 5.3 is an obvious modification of the (presented in Section 4) proof of Theorem 1.2, and it is left to the reader. □

The most general result of this kind is the following:

Theorem 5.4. *Let m, n, s be positive integers and let $F : \mathcal{M}f \to \mathcal{VB}$ be a vector bundle functor with the point property. Any regular local $\mathcal{FM}_{m,n}$-natural operator*

$$C : \mathcal{C}^\infty_M\Big(J^sY, \bigwedge^m T^*M\Big) \times \cdots \times \mathcal{C}^\infty_M\Big(J^sY, \bigwedge^m T^*M\Big) \times \mathcal{C}^\infty_M((FM)^*) \to \mathcal{C}^\infty_Y\Big(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\Big)$$

is of the form $C(\lambda_1, \dots, \lambda_k, \eta) = c_1\Lambda(\lambda_1) + \cdots + c_k\Lambda(\lambda_k)$, $c_1, \dots, c_k \in \mathbf{R}$, where Λ is the Legendre operator.

Proof. The proof of Theorem 5.4 is an obvious modification of the compilation of both the proof of Theorem 1.1 and the proof of Theorem 1.2, and it is left to the reader. □

Because of the result from [1] mentioned in Introduction, the point property of F in the above theorems is essential.

6. Final observations. Let $\mathcal{AB}_{m,n}$ denote the category of all affine bundles $A \to M$ with m-dimensional bases and n-dimensional fibres and their affine bundle isomorphisms onto open images. We have the following:

Theorem 6.1. *Let m, n, s be positive integers. Any regular local $\mathcal{AB}_{m,n}$-natural (i.e. invariant with respect to $\mathcal{AB}_{m,n}$-maps) operator*

$$C : \mathcal{C}^\infty_M\Big(J^sA, \bigwedge^m T^*M\Big) \times \mathcal{C}^\infty_M\Big(J^sA, \bigwedge^m T^*M\Big) \to \mathcal{C}^\infty_A\Big(J^{2s}A, S^sTM \otimes V^*A \otimes \bigwedge^m T^*M\Big)$$

is of the form $C(\lambda_1, \lambda_2) = c_1\Lambda(\lambda_1) + c_2\Lambda(\lambda_2)$, $c_1, c_2 \in \mathbf{R}$, where Λ is the Legendre operator.

Theorem 6.2. *Let m, n, s, p be positive integers. Any regular local $\mathcal{AB}_{m,n}$-natural operator*

$$C : \mathcal{C}^\infty_M\Big(J^sA, \bigwedge^m T^*M\Big) \times \Omega^p(M) \to \mathcal{C}^\infty_A\Big(J^sA, S^sTM \otimes V^*A \otimes \bigwedge^m T^*M\Big)$$

is of the form $C(\lambda, \eta) = c\Lambda(\lambda)$, $c \in \mathbf{R}$, where Λ is the Legendre operator.

Proof. All $\mathcal{FM}_{m,n}$-maps we used in the proofs of Theorems 1.1 and 1.2 are $\mathcal{AB}_{m,n}$-maps, except $\mathcal{FM}_{m,n}$-charts. But they may be replaced by $\mathcal{AB}_{m,n}$-charts if we consider $\mathcal{AB}_{m,n}$-natural operators. □

Clearly, the $\mathcal{AB}_{m,n}$-versions of Theorems 5.1–5.4 hold, too.

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