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ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

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GAMALIEL MORALES

On third-order Jacobsthal numbers and their bihyperbolic generalizations

Abstract. In this paper, we introduce bihyperbolic third-order Jacobsthal and third-order Jacobsthal–Lucas numbers. In other words, bihyperbolic numbers whose coefficients are consecutive third-order Jacobsthal and third-order Jacobsthal–Lucas numbers. Furthermore, we study one parameter generalizations of bihyperbolic third-order Jacobsthal and third-order Jacobsthal–Lucas numbers and relations between them.

1. Preliminary results. Let $n \geq 0$ be an integer. The nth third-order Jacobsthal number $J_n^{(3)}$ and the nth third-order Jacobsthal–Lucas number $j_n^{(3)}$ are defined recursively by

$$J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)},\ J_0^{(3)} = 0,\ J_1^{(3)} = J_2^{(3)} = 1$$

and

$$j_{n+3}^{(3)} = j_{n+2}^{(3)} + j_{n+1}^{(3)} + 2j_n^{(3)},\ j_0^{(3)} = 2,\ j_1^{(3)} = 1,\ J_2^{(3)} = 5,$$

respectively. Note that third-order Jacobsthal numbers are introduced by Cook and Bacon (see [6]) as a generalization of the Jacobsthal numbers (see [7]). The Binet type formula of these sequence have the form $J_n^{(3)} = \frac{1}{7}\left[2^{n+1} + X_n - 2X_{n+1}\right]$ and $j_n^{(3)} = \frac{1}{7}\left[2^{n+3} - 3\left(X_n - 2X_{n+1}\right)\right]$, where X_n is

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defined recursively by

$$X_{n+2} = -X_{n+1} - X_n,\ X_0 = 0,\ X_1 = 1.$$

Third-order Jacobsthal sequence has been studied in many papers, see for example [8, 9, 10, 11, 14]. Recently, Morales [12] introduced the generalized third-order Jacobsthal numbers as follows. Let $k \geq 1$ be a fixed integer, let $J_{k,n}^{(3)}$ be the nth generalized third-order Jacobsthal number defined by

$$J_{k,n+3}^{(3)} = (k-1)J_{k,n+2}^{(3)} + (k-1)J_{k,n+1}^{(3)} + kJ_{k,n}^{(3)}, \tag{1}$$

where $J_{k,0}^{(3)} = 0$, $J_{k,1}^{(3)} = 1$ and $J_{k,2}^{(3)} = k-1$. It is easily seen that $J_{2,n}^{(3)} = J_n^{(3)}$.

By analogy, we define the generalized third-order Jacobsthal–Lucas numbers in the following way. Let $k \geq 1$ be a fixed integer, let $j_{k,n}^{(3)}$ be the nth generalized third-order Jacobsthal–Lucas number defined by

$$j_{k,n+3}^{(3)} = (k-1)j_{k,n+2}^{(3)} + (k-1)j_{k,n+1}^{(3)} + kj_{k,n}^{(3)}, \tag{2}$$

where $j_{k,0}^{(3)} = 2$, $j_{k,1}^{(3)} = k-1$ and $j_{k,2}^{(3)} = k^2+1$. Moreover, we have $j_{2,n}^{(3)} = j_n^{(3)}$.

Some properties, identities and matrix generators of $J_{k,n}^{(3)}$ were given in [12]. In the next section of the paper, we use the following results.

Theorem 1.1 ([12], Theorem 2.3). *Let $n \geq 0$, $k \geq 1$ be integers. Then*

$$J_{k,n}^{(3)} = \frac{1}{k^2+k+1}\left[k^{n+1} + X_n - kX_{n+1}\right] \tag{3}$$

and

$$j_{k,n}^{(3)} = \frac{1}{k^2+k+1}\left[\left(k^2+k+2\right)k^n - (k+1)\left(X_n - kX_{n+1}\right)\right], \tag{4}$$

where $X_{n+2} = -X_{n+1} - X_n$, $X_0 = 0$ and $X_1 = 1$.

Theorem 1.2 ([12], Theorem 3.3). *Let $n \geq m \geq 0$, $k \geq 1$ be integers. Then*

$$J_{k,m+1}^{(3)}J_{k,n}^{(3)} - J_{k,m}^{(3)}J_{k,n+1}^{(3)} = \frac{1}{\sigma_k}\left[k^{n+1}X_{m+1} - k^{m+1}X_{n+1} + X_{n-m}\right] \tag{5}$$

and

$$\begin{aligned} &j_{k,m+1}^{(3)}j_{k,n}^{(3)} - j_{k,m}^{(3)}j_{k,n+1}^{(3)} \\ &\quad = \frac{(k+1)(\sigma_k+1)}{\sigma_k}\left[k^m X_{n+1} - k^n X_{m+1} + \frac{k+1}{\sigma_k+1}X_{n-m}\right], \end{aligned} \tag{6}$$

where $\sigma_k = k^2+k+1$.

Now we prove a new result on these sequences that will be used later. This result shows the relation between two consecutive terms of these sequences and their relationship with the sequence X_n.

Proposition 1.3. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we have*

$$J^{(3)}_{k,n+1} - kJ^{(3)}_{k,n} = X_{n+1} = -\frac{1}{k+1}\left[j^{(3)}_{k,n+1} - kj^{(3)}_{k,n}\right], \tag{7}$$

where X_n as in Theorem 1.1.

Proof. (By induction on n) If $n = 0$, then $J^{(3)}_{k,1} - kJ^{(3)}_{k,0} = 1 - k \cdot 0 = 1 = X_1$. Now assume that for any $n \geq 0$, we have $J^{(3)}_{k,n+1} - kJ^{(3)}_{k,n} = X_{n+1}$ and $J^{(3)}_{k,n+2} - kJ^{(3)}_{k,n+1} = X_{n+2}$. We shall show that $J^{(3)}_{k,n+3} - kJ^{(3)}_{k,n+2} = X_{n+3}$. Applying the induction's hypothesis, we obtain

$$\begin{aligned} X_{n+3} &= -X_{n+2} - X_{n+1} \\ &= -\left(J^{(3)}_{k,n+2} - kJ^{(3)}_{k,n+1}\right) - \left(J^{(3)}_{k,n+1} - kJ^{(3)}_{k,n}\right) \\ &= -J^{(3)}_{k,n+2} + (k-1)J^{(3)}_{k,n+1} + kJ^{(3)}_{k,n} \\ &= -J^{(3)}_{k,n+2} + J^{(3)}_{k,n+3} - (k-1)J^{(3)}_{k,n+2} \\ &= J^{(3)}_{k,n+3} - kJ^{(3)}_{k,n+2} \end{aligned}$$

and by induction's rule the first formula follows. The second statement is similar. □

The third-order Jacobsthal numbers and their generalizations have applications also in the theory of hypercomplex numbers. In [8], the author introduced and studied third-order Jacobsthal quaternions. In [9], the author considered the dual third-order Jacobsthal numbers and dual third-order Jacobsthal vectors. In this paper, we use third-order Jacobsthal, third-order Jacobsthal–Lucas and their generalizations in the theory of bihyperbolic numbers.

Hyperbolic numbers are two dimensional number system. Hyperbolic imaginary unit (or unipotent) is an element $h \neq \pm 1$ such that $h^2 = 1$. Bihyperbolic numbers are a generalization of hyperbolic numbers. A bihyperbolic number has the form

$$\Xi = \xi_0 + \xi_1 h_1 + \xi_2 h_2 + \xi_3 h_3, \quad \xi_l \in \mathbb{R}, \ l = 0, 1, 2, 3.$$

Note that $h_l \notin \mathbb{R}$ $(l = 1, 2, 3)$ are operators such that

$$h_1^2 = h_2^2 = h_3^2 = 1 \tag{8}$$

and

$$h_1 = h_2 h_3 = h_3 h_2, \ h_2 = h_1 h_3 = h_3 h_1, \ h_3 = h_1 h_2 = h_2 h_1. \tag{9}$$

The addition and multiplication of bihyperbolic numbers is commutative and associative, that is the set of bihyperbolic numbers is a commutative ring. For interested readers in the algebraic properties of bihyperbolic numbers, see [1] and the references cited therein. The Fibonacci numbers and their generalizations have applications also in the theory of bihyperbolic numbers. In [2, 3], Bród, Szynal-Liana and Włoch introduced and studied bihyperbolic numbers whose coefficients are consecutive Fibonacci numbers. Other interesting studies related to this type of sequences are introduced and studied in [4, 5].

Let $n \geq 0$ be an integer. The nth bihyperbolic third-order Jacobsthal number $BhJ_n^{(3)}$ and the nth bihyperbolic third-order Jacobsthal–Lucas $Bhj_n^{(3)}$ are defined by

$$BhJ_n^{(3)} = J_n^{(3)} + J_{n+1}^{(3)}h_1 + J_{n+2}^{(3)}h_2 + J_{n+3}^{(3)}h_3$$

and

$$Bhj_n^{(3)} = j_n^{(3)} + j_{n+1}^{(3)}h_1 + j_{n+2}^{(3)}h_2 + j_{n+3}^{(3)}h_3,$$

respectively, where $J_n^{(3)}$ is the nth third-order Jacobsthal number, $j_n^{(3)}$ is the nth third-order Jacobsthal–Lucas number and h_1, h_2, h_3 are units which satisfy equations (8) and (9).

Definition 1.4. The nth generalized bihyperbolic third-order Jacobsthal (or bihyperbolic third-order k-Jacobsthal) number $BhJ_{k,n}^{(3)}$ we define in the following way

$$BhJ_{k,n}^{(3)} = J_{k,n}^{(3)} + J_{k,n+1}^{(3)}h_1 + J_{k,n+2}^{(3)}h_2 + J_{k,n+3}^{(3)}h_3, \tag{10}$$

where $J_{k,n}^{(3)}$ is the nth third-order k-Jacobsthal defined in (1). By analogy, the nth generalized bihyperbolic third-order Jacobsthal–Lucas (or bihyperbolic third-order k-Jacobsthal–Lucas) number $Bhj_{k,n}^{(3)}$ we define in the following way

$$Bhj_{k,n}^{(3)} = j_{k,n}^{(3)} + j_{k,n+1}^{(3)}h_1 + j_{k,n+2}^{(3)}h_2 + j_{k,n+3}^{(3)}h_3, \tag{11}$$

where $j_{k,n}^{(3)}$ is the nth third-order k-Jacobsthal–Lucas defined in (2). For $k = 2$, we obtain $BhJ_{2,n}^{(3)} = BhJ_n^{(3)}$ and $Bhj_{2,n}^{(3)} = Bhj_n^{(3)}$.

Using the above definitions, we can write initial bihyperbolic third-order k-Jacobsthal numbers

$$\begin{aligned} BhJ_{k,0}^{(3)} &= h_1 + (k-1)h_2 + (k^2-k)h_3, \\ BhJ_{k,1}^{(3)} &= 1 + (k-1)h_1 + (k^2-k)h_2 + (k^3-k^2+1)h_3, \\ BhJ_{k,2}^{(3)} &= k-1 + (k^2-k)h_1 + (k^3-k^2+1)h_2 + (k^4-k^3+k-1)h_3, \end{aligned} \tag{12}$$

bihyperbolic third-order k-Jacobsthal–Lucas numbers

$$\begin{aligned} Bhj_{k,0}^{(3)} &= 2 + (k-1)h_1 + (k^2+1)h_2 + (k^3+k)h_3, \\ Bhj_{k,1}^{(3)} &= k-1 + (k^2+1)h_1 + (k^3+k)h_2 + (k^4+k^2-k-1)h_3, \\ Bhj_{k,2}^{(3)} &= k^2+1 + (k^3+k)h_1 + (k^4+k^2-k-1)h_2 + (k^5+k^3-k^2+1)h_3, \end{aligned} \tag{13}$$

bihyperbolic third-order Jacobsthal numbers

$$\begin{aligned} BhJ_0^{(3)} &= h_1 + h_2 + 2h_3, \\ BhJ_1^{(3)} &= 1 + h_1 + 2h_2 + 5h_3, \\ BhJ_2^{(3)} &= 1 + 2h_1 + 5h_2 + 9h_3, \end{aligned}$$

bihyperbolic third-order Jacobsthal–Lucas numbers

$$\begin{aligned} Bhj_0^{(3)} &= 2 + h_1 + 5h_2 + 10h_3, \\ Bhj_1^{(3)} &= 1 + 5h_1 + 10h_2 + 17h_3, \\ Bhj_2^{(3)} &= 5 + 10h_1 + 17h_2 + 37h_3. \end{aligned}$$

2. Main results. In this section, we present some properties of bihyperbolic third-order k-Jacobsthal and bihyperbolic third-order k-Jacobsthal–Lucas numbers.

Theorem 2.1. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we have*

$$BhJ_{k,n+3}^{(3)} = (k-1)BhJ_{k,n+2}^{(3)} + (k-1)BhJ_{k,n+1}^{(3)} + kBhJ_{k,n}^{(3)},$$

where $BhJ_{k,0}^{(3)}$, $BhJ_{k,1}^{(3)}$ and $BhJ_{k,2}^{(3)}$ are defined by (12).

Proof. By formulas (1) and (10), we have

$$\begin{aligned} &(k-1)BhJ_{k,n+2}^{(3)} + (k-1)BhJ_{k,n+1}^{(3)} + kBhJ_{k,n}^{(3)} \\ &\qquad = (k-1)\left[J_{k,n+2}^{(3)} + J_{k,n+3}^{(3)}h_1 + J_{k,n+4}^{(3)}h_2 + J_{k,n+5}^{(3)}h_3\right] \\ &\qquad\quad + (k-1)\left[J_{k,n+1}^{(3)} + J_{k,n+2}^{(3)}h_1 + J_{k,n+3}^{(3)}h_2 + J_{k,n+4}^{(3)}h_3\right] \\ &\qquad\quad + k\left[J_{k,n}^{(3)} + J_{k,n+1}^{(3)}h_1 + J_{k,n+2}^{(3)}h_2 + J_{k,n+3}^{(3)}h_3\right] \\ &\qquad = (k-1)J_{k,n+2}^{(3)} + (k-1)J_{k,n+1}^{(3)} + kJ_{k,n}^{(3)} \\ &\qquad\quad + \left[(k-1)J_{k,n+3}^{(3)} + (k-1)J_{k,n+2}^{(3)} + kJ_{k,n+1}^{(3)}\right]h_1 \\ &\qquad\quad + \left[(k-1)J_{k,n+4}^{(3)} + (k-1)J_{k,n+3}^{(3)} + kJ_{k,n+2}^{(3)}\right]h_2 \\ &\qquad\quad + \left[(k-1)J_{k,n+5}^{(3)} + (k-1)J_{k,n+4}^{(3)} + kJ_{k,n+3}^{(3)}\right]h_3 \\ &\qquad = J_{k,n+3}^{(3)} + J_{k,n+4}^{(3)}h_1 + J_{k,n+5}^{(3)}h_2 + J_{k,n+6}^{(3)}h_3 \\ &\qquad = BhJ_{k,n+3}^{(3)}, \end{aligned}$$

which proves what was requested. □

In the same way, using the formulas (2) and (11), we can prove the next result.

Theorem 2.2. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we have*

$$Bhj_{k,n+3}^{(3)} = (k-1)Bhj_{k,n+2}^{(3)} + (k-1)Bhj_{k,n+1}^{(3)} + kBhj_{k,n}^{(3)},$$

where $Bhj_{k,0}^{(3)}$, $Bhj_{k,1}^{(3)}$ and $Bhj_{k,2}^{(3)}$ are defined by equation (13).

Now we study the comparison between two consecutive terms of bihyperbolic third-order k-Jacobsthal numbers and their relation with the sequence X_n. Note that $X_{n+2} = -X_{n+1} - X_n$ for all integer $n \geq 0$.

Theorem 2.3. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we obtain*

$$BhJ_{k,n+1}^{(3)} = kBhJ_{k,n}^{(3)} + (h_2 - h_1)X_n + (h_3 - h_1 + 1)X_{n+1},$$

where X_n is defined by Theorem 1.1.

Proof. Using the relation $J_{k,n+1}^{(3)} - kJ_{k,n}^{(3)} = X_{n+1}$ in Proposition 1.3 and equation (10), we get

$$\begin{aligned} BhJ_{k,n+1}^{(3)} - kBhJ_{k,n}^{(3)} &= J_{k,n+1}^{(3)} + J_{k,n+2}^{(3)}h_1 + J_{k,n+3}^{(3)}h_2 + J_{k,n+4}^{(3)}h_3 \\ &\quad - k\left[J_{k,n}^{(3)} + J_{k,n+1}^{(3)}h_1 + J_{k,n+2}^{(3)}h_2 + J_{k,n+3}^{(3)}h_3\right] \\ &= J_{k,n+1}^{(3)} - kJ_{k,n}^{(3)} + \left[J_{k,n+2}^{(3)} - kJ_{k,n+1}^{(3)}\right]h_1 \\ &\quad + \left[J_{k,n+3}^{(3)} - kJ_{k,n+2}^{(3)}\right]h_2 + \left[J_{k,n+4}^{(3)} - kJ_{k,n+3}^{(3)}\right]h_3 \\ &= X_{n+1} + X_{n+2}h_1 + X_{n+3}h_2 + X_{n+4}h_3 \\ &= X_{n+1} + (-X_{n+1} - X_n)h_1 + X_nh_2 + X_{n+1}h_3 \\ &= (h_2 - h_1)X_n + (h_3 - h_1 + 1)X_{n+1}, \end{aligned}$$

which proves what was requested. □

Remark 2.4. Using the definition of sequence X_n in the above result, we have

$$(h_2 - h_1)X_n + (h_3 - h_1 + 1)X_{n+1} = \begin{cases} 1 - h_1 + h_3 & \text{if } n \equiv 0 \pmod 3 \\ -1 + h_2 - h_3 & \text{if } n \equiv 1 \pmod 3 \\ h_1 - h_2 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Finally, we can write Theorem 2.3 as

$$BhJ_{k,n+1}^{(3)} = \begin{cases} kBhJ_{k,n}^{(3)} + 1 - h_1 + h_3 & \text{if } n \equiv 0 \pmod 3 \\ kBhJ_{k,n}^{(3)} - 1 + h_2 - h_3 & \text{if } n \equiv 1 \pmod 3 \\ kBhJ_{k,n}^{(3)} + h_1 - h_2 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

In the same way, using the relation $j_{k,n+1}^{(3)} - kj_{k,n}^{(3)} = -(k+1)X_{n+1}$ in Proposition 1.3 and (11), we can prove the next result.

Theorem 2.5. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we have*

$$Bhj^{(3)}_{k,n+1} = kBhj^{(3)}_{k,n} - (k+1)\left[(h_2 - h_1)X_n + (h_3 - h_1 + 1)X_{n+1}\right],$$

where X_n is defined by Theorem 1.1.

Next theorems give a kind of the Binet formulas for the bihyperbolic third-order k-Jacobsthal and bihyperbolic third-order k-Jacobsthal–Lucas numbers.

Theorem 2.6. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we have*

$$BhJ^{(3)}_{k,n} = \Xi_k J^{(3)}_{k,n} + BhJ^{(3)}_{k,0} X_{n+1} - BhJ^{(3)}_{k,-1} X_n.$$

where $\Xi_k = 1 + kh_1 + k^2 h_2 + k^3 h_3$ and X_n is defined by Theorem 1.1.

Proof. Using Proposition 1.3, we have

$$\begin{aligned}
J^{(3)}_{k,n+1} &= kJ^{(3)}_{k,n} + X_{n+1},\\
J^{(3)}_{k,n+2} &= kJ^{(3)}_{k,n+1} + X_{n+2} = k^2 J^{(3)}_{k,n} + (k-1)X_{n+1} - X_n,\\
J^{(3)}_{k,n+3} &= kJ^{(3)}_{k,n+2} + X_{n+3} = k^3 J^{(3)}_{k,n} + (k^2 - k)X_{n+1} - (k-1)X_n.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
BhJ^{(3)}_{k,n} &= J^{(3)}_{k,n} + J^{(3)}_{k,n+1} h_1 + J^{(3)}_{k,n+2} h_2 + J^{(3)}_{k,n+3} h_3\\
&= J^{(3)}_{k,n} + \left[kJ^{(3)}_{k,n} + X_{n+1}\right] h_1 + \left[k^2 J^{(3)}_{k,n} + (k-1)X_{n+1} - X_n\right] h_2\\
&\quad + \left[k^3 J^{(3)}_{k,n} + (k^2 - k)X_{n+1} - (k-1)X_n\right] h_3\\
&= (1 + kh_1 + k^2 h_2 + k^3 h_3) J^{(3)}_{k,n}\\
&\quad + (h_1 + (k-1)h_2 + (k^2 - k)h_3)X_{n+1} - (h_2 + (k-1)h_3)X_n.
\end{aligned}$$

From equation (12) and $J^{(3)}_{k,-1} = 0$, we get $BhJ^{(3)}_{k,0} = h_1 + (k-1)h_2 + (k^2 - k)h_3$ and $BhJ^{(3)}_{k,-1} = h_2 + (k-1)h_3$. Finally, using the notation $\Xi_k = 1 + kh_1 + k^2 h_2 + k^3 h_3$, we obtain the desired formula. □

Theorem 2.7. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we have*

$$Bhj^{(3)}_{k,n} = \Xi_k j^{(3)}_{k,n} - (k+1)\left[BhJ^{(3)}_{k,0} X_{n+1} - BhJ^{(3)}_{k,-1} X_n\right].$$

where $\Xi_k = 1 + kh_1 + k^2 h_2 + k^3 h_3$ and X_n is defined by Theorem 1.1.

Proof. Using Proposition 1.3, equation (11) and proceeding analogously as in the proof of the previous theorem we obtain the desired formula. □

Corollary 2.8. *Let $k=2$. For any integer $n\geq 0$, we have*

$$BhJ_n^{(3)}=\Xi_2 J_n^{(3)}+(h_1+h_2+2h_3)X_{n+1}-(h_2+h_3)X_n$$

and

$$Bhj_n^{(3)}=\Xi_2 j_n^{(3)}-3\left[(h_1+h_2+2h_3)X_{n+1}-(h_2+h_3)X_n\right],$$

where $\Xi_2=1+2h_1+4h_2+8h_3$ and X_n is defined by Theorem 1.1.

For simplicity of notation let $\alpha_k=BhJ_{k,-1}^{(3)}$, $\beta_k=BhJ_{k,0}^{(3)}$ and $Z_{k,n}=\beta_k X_{n+1}-\alpha_k X_n$. Using Theorems (2.6) and (2.7), we can write

$$BhJ_{k,n}^{(3)}=\Xi_k J_{k,n}^{(3)}+Z_{k,n} \tag{14}$$

and

$$Bhj_{k,n}^{(3)}=\Xi_k j_{k,n}^{(3)}-(k+1)Z_{k,n}, \tag{15}$$

where $\Xi_k=1+kh_1+k^2h_2+k^3h_3$.

Using the Binet formulas in equations (14) and (15), we can derive the d'Ocagne identity for the bihyperbolic third-order k-Jacobsthal and bihyperbolic third-order k-Jacobsthal–Lucas numbers.

Theorem 2.9. *Let $n\geq 0$, $m\geq 0$ be integers such that $n\geq m$. Then*

$$Z_{k,m+1}Z_{k,n}-Z_{k,m}Z_{k,n+1}=(\alpha_k^2+\alpha_k\beta_k+\beta_k^2)X_{n-m}$$

and

$$\begin{aligned}BhJ_{k,m+1}^{(3)}BhJ_{k,n}^{(3)}-BhJ_{k,m}^{(3)}BhJ_{k,n+1}^{(3)}&=\frac{\Xi_k^2}{\sigma_k}\left[k^{n+1}X_{m+1}-k^{m+1}X_{n+1}+X_{n-m}\right]\\&\quad+(\alpha_k^2+\alpha_k\beta_k+\beta_k^2)X_{n-m}\\&\quad+\Xi_k\left[A_{k,m}X_{n+1}-B_{k,m}X_n\right]\\&\quad-\Xi_k\left[A_{k,n}X_{m+1}-B_{k,n}X_m\right],\end{aligned}$$

where $\Xi_k=1+kh_1+k^2h_2+k^3h_3$, $A_{k,n}=\beta_k J_{k,n+1}^{(3)}+(\alpha_k+\beta_k)J_{k,n}^{(3)}$, $B_{k,n}=\alpha_k J_{k,n+1}^{(3)}-\beta_k J_{k,n}^{(3)}$, $J_{k,n}^{(3)}$ and X_n are defined by Theorem 1.1.

Proof. By formulas $Z_{k,n}=\beta_k X_{n+1}-\alpha_k X_n$ and $Z_{k,n+1}=-(\alpha_k+\beta_k)X_{n+1}-\beta_k X_n$, we get

$$\begin{aligned}Z_{k,m+1}Z_{k,n}-Z_{k,m}Z_{k,n+1}&=\left[-(\alpha_k+\beta_k)X_{m+1}-\beta_k X_m\right]\left[\beta_k X_{n+1}-\alpha_k X_n\right]\\&\quad-\left[\beta_k X_{m+1}-\alpha_k X_m\right]\left[-(\alpha_k+\beta_k)X_{n+1}-\beta_k X_n\right]\\&=(\alpha_k^2+\alpha_k\beta_k+\beta_k^2)\left[X_{m+1}X_n-X_mX_{n+1}\right]\\&=(\alpha_k^2+\alpha_k\beta_k+\beta_k^2)X_{n-m}.\end{aligned}$$

Using equations (14) and (5) and some algebraic calculations, we obtain

$$\begin{aligned}
&BhJ^{(3)}_{k,m+1}BhJ^{(3)}_{k,n} - BhJ^{(3)}_{k,m}BhJ^{(3)}_{k,n+1}\\
&= \left[\Xi_k J^{(3)}_{k,m+1} + Z_{k,m+1}\right]\left[\Xi_k J^{(3)}_{k,n} + Z_{k,n}\right] - \left[\Xi_k J^{(3)}_{k,m} + Z_{k,m}\right]\left[\Xi_k J^{(3)}_{k,n+1} + Z_{k,n+1}\right]\\
&= \Xi_k^2\left[J^{(3)}_{k,m+1}J^{(3)}_{k,n} - J^{(3)}_{k,m}J^{(3)}_{k,n+1}\right] + Z_{k,m+1}Z_{k,n} - Z_{k,m}Z_{k,n+1}\\
&\quad + \Xi_k\left[J^{(3)}_{k,m+1}Z_{k,n} - J^{(3)}_{k,m}Z_{k,n+1} - J^{(3)}_{k,n+1}Z_{k,m} + J^{(3)}_{k,n}Z_{k,m+1}\right]\\
&= \frac{\Xi_k^2}{\sigma_k}\left[k^{n+1}X_{m+1} - k^{m+1}X_{n+1} + X_{n-m}\right]\\
&\quad + (\alpha_k^2 + \alpha_k\beta_k + \beta_k^2)X_{n-m}\\
&\quad + \Xi_k\left[A_{k,m}X_{n+1} - B_{k,m}X_n\right] - \Xi_k\left[A_{k,n}X_{m+1} - B_{k,n}X_m\right],
\end{aligned}$$

where $A_{k,n} = \beta_k J^{(3)}_{k,n+1} + (\alpha_k + \beta_k)J^{(3)}_{k,n}$ and $B_{k,n} = \alpha_k J^{(3)}_{k,n+1} - \beta_k J^{(3)}_{k,n}$, which completes the proof. □

In the same way, using equations (15) and (6), we obtain the d'Ocagne identity for the bihyperbolic third-order k-Jacobsthal–Lucas numbers.

Theorem 2.10. *Let $n \geq 0$, $m \geq 0$ be integers such that $n \geq m$. Then*

$$\begin{aligned}
Bhj^{(3)}_{k,m+1}Bhj^{(3)}_{k,n} - Bhj^{(3)}_{k,m}Bhj^{(3)}_{k,n+1}
&= \frac{\Xi_k^2(k+1)(\sigma_k+1)}{\sigma_k}\left[k^m X_{n+1} - k^n X_{m+1} + \frac{k+1}{\sigma_k+1}X_{n-m}\right]\\
&\quad + (k+1)^2(\alpha_k^2 + \alpha_k\beta_k + \beta_k^2)X_{n-m}\\
&\quad + (k+1)\Xi_k\left[C_{k,n}X_{m+1} - D_{k,n}X_m\right]\\
&\quad - (k+1)\Xi_k\left[C_{k,m}X_{n+1} - D_{k,m}X_n\right],
\end{aligned}$$

where $C_{k,n} = \beta_k j^{(3)}_{k,n+1} + (\alpha_k + \beta_k)j^{(3)}_{k,n}$, $D_{k,n} = \alpha_k j^{(3)}_{k,n+1} - \beta_k j^{(3)}_{k,n}$, $j^{(3)}_{k,n}$ and X_n are defined by Theorem 1.1.

Now, for $m = n-1$ and $n \geq 1$, we obtain Cassini identities for the bihyperbolic third-order k-Jacobsthal and bihyperbolic third-order k-Jacobsthal–Lucas numbers.

Corollary 2.11. *Let $n \geq 1$ be an integer. Then*

$$\begin{aligned}
\left[BhJ^{(3)}_{k,n}\right]^2 - BhJ^{(3)}_{k,n-1}BhJ^{(3)}_{k,n+1} &= \frac{\Xi_k^2}{\sigma_k}\left[k^{n+1}X_n - k^n X_{n+1} + 1\right]\\
&\quad + \alpha_k^2 + \alpha_k\beta_k + \beta_k^2\\
&\quad + \Xi_k\left[A_{k,n-1}X_{n+1} - B_{k,n-1}X_n\right]\\
&\quad - \Xi_k\left[A_{k,n}X_n - B_{k,n}X_{n-1}\right]
\end{aligned}$$

and

$$\begin{aligned}&\left[Bhj_{k,n}^{(3)}\right]^2-Bhj_{k,n-1}^{(3)}Bhj_{k,n+1}^{(3)}\\&\qquad=\frac{\Xi_k^2(k+1)(\sigma_k+1)}{\sigma_k}\left[k^{n-1}X_{n+1}-k^nX_n+\frac{k+1}{\sigma_k+1}\right]\\&\qquad\quad+(k+1)^2(\alpha_k^2+\alpha_k\beta_k+\beta_k^2)\\&\qquad\quad+(k+1)\Xi_k\left[C_{k,n}X_n-D_{k,n}X_{n-1}\right]\\&\qquad\quad-(k+1)\Xi_k\left[C_{k,n-1}X_{n+1}-D_{k,n-1}X_n\right].\end{aligned}$$

Now we give ordinary generating functions for the bihyperbolic third-order k-Jacobsthal and bihyperbolic third-order k-Jacobsthal–Lucas numbers.

Theorem 2.12. *Let $k\geq 1$ be a fixed integer. The generating function for the bihyperbolic third-order k-Jacobsthal number sequence $\{BhJ_{k,n}^{(3)}\}$ is given by*

$$G(\lambda)=G\left(BhJ_{k,n}^{(3)};\lambda\right)=\frac{\left\{\begin{array}{c}h_1+(k-1)h_2+(k^2-k)h_3\\+[1+(k-1)h_2+(k^2-k+1)h_3]\lambda\\+[kh_2+(k^2-k)h_3]\lambda^2\end{array}\right\}}{1-(k-1)\lambda-(k-1)\lambda^2-k\lambda^3}.$$

Proof. Assume that the generating function of the bihyperbolic third-order k-Jacobsthal number sequence $\{BhJ_{k,n}^{(3)}\}$ has the form $G(\lambda)=BhJ_{k,0}^{(3)}+BhJ_{k,1}^{(3)}\lambda+BhJ_{k,2}^{(3)}\lambda^2+\cdots$. Then, after some algebraic calculations

$$\begin{aligned}&\left[1-(k-1)\lambda-(k-1)\lambda^2-k\lambda^3\right]G(\lambda)\\&=BhJ_{k,0}^{(3)}+BhJ_{k,1}^{(3)}\lambda+BhJ_{k,2}^{(3)}\lambda^2+BhJ_{k,3}^{(3)}\lambda^3\cdots\\&\quad-(k-1)BhJ_{k,0}^{(3)}\lambda-(k-1)BhJ_{k,1}^{(3)}\lambda^2-(k-1)BhJ_{k,2}^{(3)}\lambda^3-\cdots\\&\quad-(k-1)BhJ_{k,0}^{(3)}\lambda^2-(k-1)BhJ_{k,1}^{(3)}\lambda^3-\cdots\\&\quad-kBhJ_{k,0}^{(3)}\lambda^3-\cdots\\&=BhJ_{k,0}^{(3)}+(BhJ_{k,1}^{(3)}-(k-1)BhJ_{k,0}^{(3)})\lambda\\&\quad+(BhJ_{k,2}^{(3)}-(k-1)BhJ_{k,1}^{(3)}-(k-1)BhJ_{k,0}^{(3)})\lambda^2,\end{aligned}$$

since $BhJ_{k,n+3}^{(3)}=(k-1)BhJ_{k,n+2}^{(3)}+(k-1)BhJ_{k,n+1}^{(3)}-kBhJ_{k,n}^{(3)}$ and the coefficients of λ^n for $n\geq 3$ are equal to zero. Furthermore, we can write $BhJ_{k,0}^{(3)}=h_1+(k-1)h_2+(k^2-k)h_3$, $BhJ_{k,1}^{(3)}-(k-1)BhJ_{k,0}^{(3)}=1+(k-1)h_2+(k^2-k+1)h_3$ and $BhJ_{k,2}^{(3)}-(k-1)BhJ_{k,1}^{(3)}-(k-1)BhJ_{k,0}^{(3)}=kh_2+(k^2-k)h_3$, which completes the proof. □

Theorem 2.13. *Let $k \geq 1$ be a fixed integer. The generating function for the bihyperbolic third-order k-Jacobsthal–Lucas number sequence $\{Bhj_{k,n}^{(3)}\}$ is given by*

$$G\left(Bhj_{k,n}^{(3)};\lambda\right) = \frac{\left\{\begin{array}{c} 2+(k-1)h_1+(k^2+1)h_2+(k^3+k)h_3 \\ +[-k+1+2kh_1+(k^2+1)h_2+(k^1-1)h_3]\lambda \\ +[2+2kh_1+(k^2-k)h_2+(k^3+k)h_3]\lambda^2 \end{array}\right\}}{1-(k-1)\lambda-(k-1)\lambda^2-k\lambda^3}.$$

Proof. This proof is similar to the proof of Theorem 2.12. In this case, we have $Bhj_{k,0}^{(3)}=2+(k-1)h_1+(k^2+1)h_2+(k^3+k)h_3$, $Bhj_{k,1}^{(3)}-(k-1)Bhj_{k,0}^{(3)}=-k+1+2kh_1+(k^2+1)h_2+(k^3-1)h_3$ and $Bhj_{k,2}^{(3)}-(k-1)Bhj_{k,1}^{(3)}-(k-1)Bhj_{k,0}^{(3)}=2+2kh_1+(k^2-k)h_2+(k^3+k)h_3$. $\square$

Remark 2.14. Let $k=2$, the generating function $G\big(BhJ_n^{(3)};\lambda\big)$ for the bihyperbolic third-order Jacobsthal number sequence $\{BhJ_n^{(3)}\}_{n\geq 0}$ is

$$G\left(BhJ_n^{(3)};\lambda\right) = \frac{h_1+h_2+2h_3+[1+h_2+3h_3]\lambda+[2h_2+2h_3]\lambda^2}{1-\lambda-\lambda^2-2\lambda^3}.$$

Also, the generating function $G\big(Bhj_n^{(3)};\lambda\big)$ for the bihyperbolic third-order Jacobsthal–Lucas number sequence $\{Bhj_n^{(3)}\}_{n\geq 0}$ is

$$G\left(Bhj_n^{(3)};\lambda\right) = \frac{\left\{\begin{array}{c} 2+h_1+5h_2+10h_3 \\ +[-1+4h_1+5h_2+7h_3]\lambda+[2+4h_1+2h_2+10h_3]\lambda^2 \end{array}\right\}}{1-\lambda-\lambda^2-2\lambda^3}.$$

Finally, we will give a matrix representation of the bihyperbolic numbers defined above. Let us consider the following matrix

$$\mathrm{N}_k^{(3)} = \begin{bmatrix} BhJ_{k,1}^{(3)} & BhJ_{k,2}^{(3)}-(k-1)BhJ_{k,1}^{(3)} & kBhJ_{k,0}^{(3)} \\ BhJ_{k,0}^{(3)} & BhJ_{k,1}^{(3)}-(k-1)BhJ_{k,0}^{(3)} & kBhJ_{k,-1}^{(3)} \\ BhJ_{k,-1}^{(3)} & BhJ_{k,0}^{(3)}-(k-1)BhJ_{k,-1}^{(3)} & kBhJ_{k,-2}^{(3)} \end{bmatrix},$$

where $BhJ_{k,-1}^{(3)}=h_2+(k-1)h_3$ and $BhJ_{k,-2}^{(3)}=\frac{1}{k}+h_3$.

It is easy to see that for $n\geq 0$ it holds

$$\left[\mathrm{M}_k^{(3)}\right]^n = \begin{bmatrix} k-1 & k-1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} J_{k,n+1}^{(3)} & TJ_{k,n+1}^{(3)} & kJ_{k,n}^{(3)} \\ J_{k,n}^{(3)} & TJ_{k,n}^{(3)} & kJ_{k,n-1}^{(3)} \\ J_{k,n-1}^{(3)} & TJ_{k,n-1}^{(3)} & kJ_{k,n-2}^{(3)} \end{bmatrix},$$

where $TJ_{k,n}^{(3)}=J_{k,n+1}^{(3)}-(k-1)J_{k,n}^{(3)}$, $J_{k,-1}^{(3)}=0$ and $J_{k,-2}^{(3)}=\frac{1}{k}$. Then, we consider the following theorem.

Theorem 2.15. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we obtain*

$$\mathrm{N}_k^{(3)} \cdot \left[M_k^{(3)}\right]^n = \begin{bmatrix} BhJ_{k,n+1}^{(3)} & ThJ_{k,n+1}^{(3)} & kBhJ_{k,n}^{(3)} \\ BhJ_{k,n}^{(3)} & ThJ_{k,n}^{(3)} & kBhJ_{k,n-1}^{(3)} \\ BhJ_{k,n-1}^{(3)} & ThJ_{k,n-1}^{(3)} & kBhJ_{k,n-2}^{(3)} \end{bmatrix},$$

where $ThJ_{k,n}^{(3)} = BhJ_{k,n+1}^{(3)} - (k-1)BhJ_{k,n}^{(3)}$.

Proof. (By induction on n) If $n = 0$, then assuming that the matrix to power 0 is the identity matrix the result is obvious. Now, suppose that for $n \geq 0$ it holds

$$N_k^{(3)} \cdot \left[\mathrm{M}_k^{(3)}\right]^n = \begin{bmatrix} BhJ_{k,n+1}^{(3)} & ThJ_{k,n+1}^{(3)} & kBhJ_{k,n}^{(3)} \\ BhJ_{k,n}^{(3)} & ThJ_{k,n}^{(3)} & kBhJ_{k,n-1}^{(3)} \\ BhJ_{k,n-1}^{(3)} & ThJ_{k,n-1}^{(3)} & kBhJ_{k,n-2}^{(3)} \end{bmatrix}.$$

By simple calculations, using induction's hypothesis we have

$$\begin{aligned} \mathrm{N}_k^{(3)} \cdot \left[\mathrm{M}_k^{(3)}\right]^{n+1} &= \mathrm{N}_k^{(3)} \cdot \left[\mathrm{M}_k^{(3)}\right]^n \cdot \mathrm{M}_k^{(3)} \\ &= \begin{bmatrix} BhJ_{k,n+1}^{(3)} & ThJ_{k,n+1}^{(3)} & kBhJ_{k,n}^{(3)} \\ BhJ_{k,n}^{(3)} & ThJ_{k,n}^{(3)} & kBhJ_{k,n-1}^{(3)} \\ BhJ_{k,n-1}^{(3)} & ThJ_{k,n-1}^{(3)} & kBhJ_{k,n-2}^{(3)} \end{bmatrix} \cdot \begin{bmatrix} k-1 & k-1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (k-1)BhJ_{k,n+1}^{(3)} + ThJ_{k,n+1}^{(3)} & ThJ_{k,n+2}^{(3)} & kBhJ_{k,n+1}^{(3)} \\ (k-1)BhJ_{k,n}^{(3)} + ThJ_{k,n}^{(3)} & ThJ_{k,n+1}^{(3)} & kBhJ_{k,n}^{(3)} \\ (k-1)BhJ_{k,n-1}^{(3)} + ThJ_{k,n-1}^{(3)} & ThJ_{k,n}^{(3)} & kBhJ_{k,n-1}^{(3)} \end{bmatrix} \\ &= \begin{bmatrix} BhJ_{k,n+2}^{(3)} & ThJ_{k,n+2}^{(3)} & kBhJ_{k,n+1}^{(3)} \\ BhJ_{k,n+1}^{(3)} & ThJ_{k,n+1}^{(3)} & kBhJ_{k,n}^{(3)} \\ BhJ_{k,n}^{(3)} & ThJ_{k,n}^{(3)} & kBhJ_{k,n-1}^{(3)} \end{bmatrix}, \end{aligned}$$

where $ThJ_{k,n}^{(3)} = BhJ_{k,n+1}^{(3)} - (k-1)BhJ_{k,n}^{(3)}$, which completes the proof. □

In addition, we can also verify the behavior of the following determinants indicated by

$$\det\left[\left(\mathrm{M}_k^{(3)}\right)^n\right] = \det \begin{bmatrix} J_{k,n+1}^{(3)} & TJ_{k,n+1}^{(3)} & kJ_{k,n}^{(3)} \\ J_{k,n}^{(3)} & TJ_{k,n}^{(3)} & kJ_{k,n-1}^{(3)} \\ J_{k,n-1}^{(3)} & TJ_{k,n-1}^{(3)} & kJ_{k,n-2}^{(3)} \end{bmatrix},$$

$$\det\left[\mathrm{N}_k^{(3)}\cdot\left(\mathrm{M}_k^{(3)}\right)^n\right]=\det\begin{bmatrix} BhJ_{k,n+1}^{(3)} & ThJ_{k,n+1}^{(3)} & kBhJ_{k,n}^{(3)} \\ BhJ_{k,n}^{(3)} & ThJ_{k,n}^{(3)} & kBhJ_{k,n-1}^{(3)} \\ BhJ_{k,n-1}^{(3)} & ThJ_{k,n-1}^{(3)} & kBhJ_{k,n-2}^{(3)} \end{bmatrix},$$

where $TJ_{k,n}^{(3)}$ and $ThJ_{k,n}^{(3)}$ as in Theorem 2.15.

Corollary 2.16. *For any integer $n\geq 0$, we obtain*

$$\det\left[\left(\mathrm{M}_k^{(3)}\right)^n\right]=k^n, \tag{16}$$

$$\det\left[\mathrm{N}_k^{(3)}\cdot\left(\mathrm{M}_k^{(3)}\right)^n\right]=\det\left[\left(\mathrm{M}_k^{(3)}\right)^n\cdot\mathrm{N}_k^{(3)}\right]=k^n\det\left[\mathrm{N}_k^{(3)}\right]. \tag{17}$$

Remark 2.17. Note that multiplication of bihyperbolic numbers is commutative and determinant properties can be used. In this sense, calculating determinants in Theorem 2.15, we can also obtain a cubic identity for the bihyperbolic third-order k-Jacobsthal and bihyperbolic third-order k-Jacobsthal–Lucas numbers. Also, using algebraic operations and matrix algebra could give many other interesting identities of these bihyperbolic numbers.

3. Conclusions. In this paper, we introduce one-parameter generalization of bihyperbolic third-order Jacobsthal numbers and bihyperbolic third-order Jacobsthal–Lucas numbers. Furthermore, we study some properties of them, among others the Binet formula, Cassini and d'Ocagne identities. Moreover, we give the generating function and special relations between them. The presented results are generalizations of the dual and gaussian third-order Jacobsthal numbers studied in [9, 13] following the ideas of Bród, Szynal-Liana and Włoch in [2, 3]. An interested reader could further generalize this sequence, considering for example arbitrary initial conditions.

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Gamaliel Morales
Instituto de Matemáticas
Pontificia Universidad Católica de Valparaíso
Blanco Viel 596, Valparaíso
Chile
e-mail: `gamaliel.cerda.m@mail.pucv.cl`

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