

ANNA BEDNARSKA

On lifts of projectable-projectable classical linear connections to the cotangent bundle

ABSTRACT. We describe all $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators $D: Q_{proj-proj}^\tau \rightsquigarrow QT^*$ transforming projectable-projectable classical torsion-free linear connections ∇ on fibred-fibred manifolds Y into classical linear connections $D(\nabla)$ on cotangent bundles T^*Y of Y . We show that this problem can be reduced to finding $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators $D: Q_{proj-proj}^\tau \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$ for $p = 2, q = 1$ and $p = 3, q = 0$.

1. Basic definitions and examples. A fibred-fibred manifold Y is any commutative diagram

$$\begin{array}{ccc} Y = Y_1 & \xrightarrow{p_{12}} & Y_2 \\ p_{13} \downarrow & & \downarrow p_{24} \\ Y_3 & \xrightarrow{p_{34}} & Y_4 \end{array}$$

where maps $p_{12}, p_{13}, p_{24}, p_{34}$ are surjective submersions and an induced map $Y_1 \rightarrow Y_2 \times_{Y_4} Y_3$, $y \mapsto (p_{12}(y), p_{13}(y))$ is a surjective submersion. A fibred-fibred manifold has dimension (m_1, m_2, n_1, n_2) if $\dim Y_1 = m_1 + m_2 + n_1 + n_2$, $\dim Y_2 = m_1 + m_2$, $\dim Y_3 = m_1 + n_1$, $\dim Y_4 = m_1$. For two fibred-fibred manifolds Y, \tilde{Y} of the same dimension (m_1, m_2, n_1, n_2) , a morphism $f: Y \rightarrow \tilde{Y}$ is a quadruple of local diffeomorphisms $f_1: Y_1 \rightarrow \tilde{Y}_1, f_2: Y_2 \rightarrow$

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$\tilde{Y}_2, f_3: Y_3 \rightarrow \tilde{Y}_3, f_4: Y_4 \rightarrow \tilde{Y}_4$ such that all squares of the cube in question are commutative, [2], [7].

All fibred-fibred manifolds of the given dimension (m_1, m_2, n_1, n_2) and all their morphisms form the category which we denote by $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$.

Every object from the category $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ is locally isomorphic to the standard fibred-fibred manifold

$$\begin{array}{ccc} \mathbb{R}^{m_1, m_2, n_1, n_2} = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} & \longrightarrow & \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \\ \downarrow & & \downarrow \\ \mathbb{R}^{m_1} \times \mathbb{R}^{n_1} & \longrightarrow & \mathbb{R}^{m_1} \end{array}$$

where arrows are obvious projections.

A classical linear connection ∇ on a fibred-fibred manifold Y is a tangent bundle homothety invariant section $\nabla: TY \rightarrow J^1TY$ of the 1-jet prolongation $J^1TY \rightarrow TY$ of the tangent bundle TY . Recall that a classical linear connection ∇ on Y is called a projectable-projectable linear connection on a fibred-fibred manifold Y if there exist classical linear connections $\nabla_2, \nabla_3, \nabla_4$ on Y_2, Y_3, Y_4 , respectively, such that the connection ∇ projects into ∇_2 and ∇_3 by maps p_{12} and p_{13} , respectively, and connections ∇_2 and ∇_3 project into ∇_4 by maps p_{24} and p_{34} , respectively, [4], [1].

A classical linear connection $\nabla: TY \rightarrow J^1TY$ on Y determines the corresponding covariant derivative $\nabla: \mathfrak{X}(Y) \times \mathfrak{X}(Y) \rightarrow \mathfrak{X}(Y)$ of vector fields on Y satisfying the additional projectability-projectability condition.

We say that a classical linear connection ∇ on a fibred-fibred manifold Y is torsion-free if the torsion tensor $T(X, Y)$ of ∇ vanishes, i.e. $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0$.

In the present paper we consider a problem of constructing of a classical linear connection $D(\nabla)$ on the cotangent bundle T^*Y of Y by means of a projectable-projectable classical torsion-free linear connection ∇ on an (m_1, m_2, n_1, n_2) -dimensional fibred-fibred manifold Y . To this aim we will consider a characterization of $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators $D: Q_{proj-proj}^\tau \rightsquigarrow QT^*$ corresponding to above constructions.

A similar problem in the case of usual n -dimensional manifolds M and classical linear connections ∇ (not necessarily torsion-free) was studied by M. Kureš [6] and it was extended to $\otimes^k T^*M$ in [5].

We will formulate definitions of natural operators which can be treated as special cases of the general concept of natural operators from [3].

Definition 1. An $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator $D: Q_{proj-proj}^\tau \rightsquigarrow QT^*$ transforming projectable-projectable classical torsion-free linear connections ∇ on (m_1, m_2, n_1, n_2) -dimensional fibred-fibred manifolds Y into classical linear connections $D(\nabla)$ on T^*Y is a family $D = (D_Y)$ of $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -invariant regular operators

$$D_Y: Q_{proj-proj}^\tau(Y) \rightarrow Q(T^*Y)$$

for any fibred-fibred manifold Y of the dimension (m_1, m_2, n_1, n_2) , where $Q_{proj-proj}^\tau(Y)$ is the set of all projectable-projectable classical torsion-free linear connections on the fibred-fibred manifold Y and $Q(T^*Y)$ is the set of all classical linear connections (not necessarily torsion-free) on T^*Y . The $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -invariance of (the operator) D means that if any projectable-projectable classical torsion-free linear connections $\nabla \in Q_{proj-proj}^\tau(Y)$, $\tilde{\nabla} \in Q_{proj-proj}^\tau(\tilde{Y})$ are φ -related by an $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -invariant map $\varphi: Y \rightarrow \tilde{Y}$ (i.e. $J^1T\varphi \circ \nabla = \tilde{\nabla} \circ T\varphi$) then induced classical linear connections $D(\nabla) \in Q(T^*Y)$ and $D(\tilde{\nabla}) \in Q(T^*\tilde{Y})$ are $T^*\varphi$ -related by $T^*\varphi: T^*Y \rightarrow T^*\tilde{Y}$ (i.e. $J^1T(T^*\varphi) \circ D(\nabla) = D(\tilde{\nabla}) \circ T(T^*\varphi)$), where $T^*\varphi$ is a cotangent map to φ .

The regularity of D means that D transforms smoothly parameterized families of projectable-projectable classical torsion-free linear connections into smoothly parameterized families of classical linear connections.

Example 1. An example of $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator

$$D: Q_{proj-proj}^\tau \rightsquigarrow QT^*$$

is a family $D^{T^*} = (D_Y^{T^*})$ of operators

$$D_Y^{T^*}: Q_{proj-proj}^\tau(Y) \rightarrow Q(T^*Y)$$

given by the formula $D_Y^{T^*}(\nabla) = \nabla^{T^*Y}$, where $Y \in Obj(\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2})$, $\nabla \in Q_{proj-proj}^\tau(Y)$ and ∇^{T^*Y} is a horizontal lift of ∇ on Y to the cotangent bundle T^*Y .

We define a horizontal lift ∇^{T^*Y} of a projectable-projectable classical torsion-free linear connection ∇ to the cotangent bundle T^*Y as

$$\nabla^{T^*Y} = \nabla^C - R_Y^V(\nabla),$$

where ∇^C is the complete lift of ∇ and $R_Y^V(\nabla)$ means the vertical lift of the curvature tensor $R_Y(\nabla)$ of ∇ , [8].

Definition 2. An $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator

$$D: Q_{proj-proj}^\tau \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$$

transforming projectable-projectable classical torsion-free linear connections ∇ on (m_1, m_2, n_1, n_2) -dimensional fibred-fibred manifolds Y into fibred maps $D(\nabla): T^*Y \rightarrow \otimes^p T^*Y \otimes \otimes^q TY$ covering the identity id_Y is a family of $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -invariant regular operators

$$D = (D_Y): Q_{proj-proj}^\tau(Y) \rightarrow C_Y^\infty(T^*Y, \otimes^p T^*Y \otimes \otimes^q TY)$$

defined for any (m_1, m_2, n_1, n_2) -dimensional fibred-fibred manifold Y .

The $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -invariance of D means that if two projectable-projectable classical torsion-free linear connections $\nabla \in Q_{proj-proj}^\tau(Y)$ and $\tilde{\nabla} \in Q_{proj-proj}^\tau(\tilde{Y})$ are φ -related by an $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -map $\varphi: Y \rightarrow \tilde{Y}$

then induced maps $D(\nabla): T^*Y \rightarrow \otimes^p T^*Y \otimes \otimes^q TY$ and $D(\tilde{\nabla}): T^*\tilde{Y} \rightarrow \otimes^p T^*\tilde{Y} \otimes \otimes^q T\tilde{Y}$ are φ -related, i.e. the following diagram is commutative

$$\begin{array}{ccc} T^*Y & \xrightarrow{D(\nabla)} & \otimes^p T^*Y \otimes \otimes^q TY \\ T^*\varphi \downarrow & & \downarrow \otimes^p T^*\varphi \otimes \otimes^q T\varphi \\ T^*\tilde{Y} & \xrightarrow{D(\tilde{\nabla})} & \otimes^p T^*\tilde{Y} \otimes \otimes^q T\tilde{Y} \end{array}$$

where $T\varphi: TY \rightarrow T\tilde{Y}$ is a tangent map to $\varphi: Y \rightarrow \tilde{Y}$ and $T^*\varphi: T^*Y \rightarrow T^*\tilde{Y}$ is a cotangent map to φ .

Example 2. An example of $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator

$$D: Q_{proj-proj}^\tau \rightsquigarrow (T^*, \otimes^3 T^*)$$

is a family of operators $D^1 = (D_Y^1)$,

$$D_Y^1: Q_{proj-proj}^\tau(Y) \rightarrow C_Y^\infty(T^*Y, \otimes^3 T^*Y),$$

$D^1(\nabla): T^*Y \rightarrow \otimes^3 T^*Y$ given by $D_Y^1(\nabla)(\omega) = \omega \otimes \omega \otimes \omega$, where $\omega \in T_y^*Y$, $y \in Y$, $Y \in \text{Obj}(\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2})$, $\nabla \in Q_{proj-proj}^\tau(Y)$.

Another example of $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator

$$D: Q_{proj-proj}^\tau \rightsquigarrow (T^*, \otimes^3 T^*)$$

is a family of operators $D^2 = (D_Y^2)$,

$$D_Y^2: Q_{proj-proj}^\tau(Y) \rightarrow C_Y^\infty(T^*Y, \otimes^3 T^*Y),$$

$D^2(\nabla): T^*Y \rightarrow \otimes^3 T^*Y$ given by $D_Y^2(\nabla)(\omega) = \langle R_y(\nabla), \omega \rangle$, where $R(\nabla)$ is the curvature tensor of ∇ , $\omega \in T_y^*Y$, $y \in Y$, $Y \in \text{Obj}(\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2})$, $\nabla \in Q_{proj-proj}^\tau(Y)$.

Example 3. An example of $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator

$$D: Q_{proj-proj}^\tau \rightsquigarrow (T^*, \otimes^2 T^* \otimes T)$$

is a family of operators $D^3 = (D_Y^3)$,

$$D_Y^3: Q_{proj-proj}^\tau(Y) \rightarrow C_Y^\infty(T^*Y, \otimes^2 T^*Y \otimes TY),$$

$D^3(\nabla): T^*Y \rightarrow \otimes^2 T^*Y \otimes TY$ given by $\langle D_Y^3(\nabla)(\omega), v_1 \otimes v_2 \rangle = \langle \omega, v_1 \rangle v_2$, where $\omega \in T_y^*Y$, $v_1, v_2 \in T_y Y$, $y \in Y$, $Y \in \text{Obj}(\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2})$, $\nabla \in Q_{proj-proj}^\tau(Y)$.

2. Some lemmas. The following lemma shows that the description of $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators $D: Q_{proj-proj}^\tau \rightsquigarrow QT^*$ can be replaced by the description of $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators $D: Q_{proj-proj}^\tau \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$.

Lemma 1. *There exists a bijection between the set of $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators $D: Q_{proj-proj}^\tau \rightsquigarrow QT^*$ and the set of sequences $(D_i)_{i=1, \dots, 8}$ consisting of $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators of the following forms:*

$$\begin{aligned} D_1: Q_{proj-proj}^\tau &\rightsquigarrow (T^*, T^* \otimes T^* \otimes T) \\ D_2: Q_{proj-proj}^\tau &\rightsquigarrow (T^*, T^* \otimes T^* \otimes T^*) \\ D_3: Q_{proj-proj}^\tau &\rightsquigarrow (T^*, T \otimes T^* \otimes T) \\ D_4: Q_{proj-proj}^\tau &\rightsquigarrow (T^*, T \otimes T^* \otimes T^*) \\ D_5: Q_{proj-proj}^\tau &\rightsquigarrow (T^*, T^* \otimes T \otimes T) \\ D_6: Q_{proj-proj}^\tau &\rightsquigarrow (T^*, T^* \otimes T \otimes T^*) \\ D_7: Q_{proj-proj}^\tau &\rightsquigarrow (T^*, T \otimes T \otimes T) \\ D_8: Q_{proj-proj}^\tau &\rightsquigarrow (T^*, T \otimes T \otimes T^*). \end{aligned}$$

Proof. Let $\nabla \in Q_{proj-proj}^\tau(Y)$ be a projectable-projectable classical torsion-free linear connection on an (m_1, m_2, n_1, n_2) -dimensional fibred-fibred manifold Y . Let $v \in T_y^*Y$, $y \in Y$.

The connection ∇ yields a decomposition of the tangent space $T_v T^*Y$ of T^*Y at v of the form

$$T_v T^*Y = H_v^\nabla \oplus V_v T^*Y,$$

where H_v^∇ is a ∇ -horizontal subspace and $V_v T^*Y$ is a vertical subspace.

We have an isomorphism $H_v^\nabla \cong T_y Y$ by the restriction of the differential $T_v \pi: T_v T^*Y \rightarrow T_y Y$ of the cotangent bundle projection $\pi: T^*Y \rightarrow Y$ to H_v^∇ . Moreover, we have an isomorphism $V_v T^*Y \cong T_y^*Y$ by the standard isomorphism

$$T_y^*Y \ni \omega \rightarrow \left. \frac{d}{dt} \right|_0 (v + t\omega) \in T_v T^*Y = V_v T^*Y.$$

Thus we have a decomposition

$$T_v T^*Y \cong T_y Y \oplus T_y^*Y$$

canonically depending on ∇ .

Consequently, we have a linear isomorphism

$$T_v^* T^*Y \otimes T_v^* T^*Y \otimes T_v T^*Y \cong (T_y Y \oplus T_y^*Y)^* \otimes (T_y Y \oplus T_y^*Y)^* \otimes (T_y Y \oplus T_y^*Y)$$

canonically depending on ∇ .

We have an isomorphism

$$(T_y Y \oplus T_y^*Y)^* \cong T_y^*Y \oplus T_y Y$$

by standard identifications

$$(V \oplus W)^* = V^* \oplus W^* \quad \text{and} \quad V^{**} = V,$$

from linear algebra.

Thus we have the following linear isomorphism

$$\begin{aligned} T_v^* T^* Y \otimes T_v^* T^* Y \otimes T_v T^* Y &\cong (T_y^* Y \otimes T_y^* Y \otimes T_y Y) \oplus (T_y^* Y \otimes T_y^* Y \otimes T_y^* Y) \\ &\oplus (T_y Y \otimes T_y^* Y \otimes T_y Y) \oplus (T_y Y \otimes T_y^* Y \otimes T_y^* Y) \oplus (T_y^* Y \otimes T_y Y \otimes T_y Y) \\ &\oplus (T_y^* Y \otimes T_y Y \otimes T_y^* Y) \oplus (T_y Y \otimes T_y Y \otimes T_y Y) \oplus (T_y Y \otimes T_y Y \otimes T_y^* Y) \end{aligned}$$

canonically depending on ∇ .

Using the above isomorphism, for any $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator $D: Q_{proj-proj}^T \rightsquigarrow QT^*$, we can define a sequence of eight $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators D_1, \dots, D_8 such as in Lemma 1, taking

$$(1) \quad (D_1(\nabla)(v), \dots, D_8(\nabla)(v)) := (D(\nabla) - \nabla^{T^*})(v)$$

for any $\nabla \in Q_{proj-proj}^T(Y)$, $Y \in Obj(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2})$, $v \in T_y^* Y$, $y \in Y$, where ∇^{T^*} is the horizontal lift of ∇ to $T^* Y$.

The difference $D(\nabla) - \nabla^{T^*}$ of linear connections $D(\nabla)$ and ∇^{T^*} means a tensor field of type $T^* \otimes T^* \otimes T$ on $T^* Y$.

Above relation (1) makes sense because it holds $(D(\nabla) - \nabla^{T^*})(v) \in T_v^* T^* Y \otimes T_v^* T^* Y \otimes T_v T^* Y$ and $(D_1(\nabla)(v), \dots, D_8(\nabla)(v)) \in ((T_y^* Y \otimes T_y^* Y \otimes T_y Y) \oplus \dots \oplus (T_y Y \otimes T_y Y \otimes T_y^* Y)) \cong T_v^* T^* Y \otimes T_v^* T^* Y \otimes T_v T^* Y$, where \cong is a linear isomorphism canonically depending on ∇ describing above.

It is obvious that an assignment $D \mapsto (D_i)_{i=1, \dots, 8}$ yields the bijection from Lemma 1. \square

Note that the description of natural operators D_1 , D_4 and D_6 from Lemma 1 can be reduced to the description of operators of type D_1 since by obviously linear isomorphisms obtaining by permutations of factors

$$T_y^* Y \otimes T_y^* Y \otimes T_y Y \cong T_y Y \otimes T_y^* Y \otimes T_y^* Y \cong T_y^* Y \otimes T_y Y \otimes T_y^* Y$$

for any $Y \in Obj(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2})$ and $y \in Y$ we have

Lemma 2. *There exists the bijection between the set of $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators $D_1: Q_{proj-proj}^T \rightsquigarrow (T^*, T^* \otimes T^* \otimes T)$ and the set of $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators $D_4: Q_{proj-proj}^T \rightsquigarrow (T^*, T \otimes T^* \otimes T^*)$.*

Similarly, there exists the bijection between the set of $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators $D_1: Q_{proj-proj}^T \rightsquigarrow (T^, T^* \otimes T^* \otimes T)$ and the set of $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators $D_6: Q_{proj-proj}^T \rightsquigarrow (T^*, T^* \otimes T \otimes T^*)$.*

Proof. The first bijection is of the form $D_1 \mapsto D_4$, where $D_4(\nabla)(v) := D_1(\nabla)(v)$, $v \in T_y^* Y$, $y \in Y$, $\nabla \in Q_{proj-proj}^T(Y)$ modulo the identification $T_y^* Y \otimes T_y^* Y \otimes T_y Y \cong T_y Y \otimes T_y^* Y \otimes T_y^* Y$ of the form $\omega_1 \otimes \omega_2 \otimes \omega \mapsto \omega \otimes \omega_1 \otimes \omega_2$.

The second bijection is analogous. \square

Moreover, we show that $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operators D_3, D_5, D_7 and D_8 from Lemma 1 are zero. It holds the following general fact.

Lemma 3. *Let $p < q$. Then every $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operator*

$$D: Q_{proj-proj}^\tau \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$$

is zero.

Proof. Let $\nabla \in Q_{proj-proj}^\tau(Y)$, $Y \in Obj(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$, $v \in T_y^*Y$, $y \in Y$. We have to show that $D(\nabla)(v) = 0 \in \otimes^p T_y^*Y \otimes \otimes^q T_yY$. By the $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -invariance of D with respect to $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -charts we can assume $Y = \mathbb{R}^{m_1,m_2,n_1,n_2}$, $y = (0, 0, 0, 0) \in \mathbb{R}^{m_1+m_2+n_1+n_2}$.

Then using the invariance of D with respect to $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -maps (homotheties)

$$\frac{1}{t}id: \mathbb{R}^{m_1,m_2,n_1,n_2} \rightarrow \mathbb{R}^{m_1,m_2,n_1,n_2} \quad \text{for } t \neq 0,$$

we get the condition

$$D(\nabla)(v) = \left(\frac{1}{t}\right)^{q-p} D\left(\left(\frac{1}{t}id\right)_* \nabla\right)(tv), \quad t \neq 0.$$

But the family (∇_t) of projectable-projectable classical torsion-free linear connections given by

$$\nabla_t := \begin{cases} \left(\frac{1}{t}id\right)_* \nabla, & t \neq 0 \\ \nabla_0, & t = 0, \end{cases}$$

where ∇_0 is the flat torsion-free linear connection (i.e. with zero Christoffel symbols), is smoothly parameterized because of the fact that ∇_t has Christoffel symbols of the form $t \cdot \Gamma_{bc}^a(tx)$ at the chart $id_{\mathbb{R}^{m_1,m_2,n_1,n_2}}$, where $\Gamma_{bc}^a(x)$ are the Christoffel symbols for ∇ .

Thus using the regularity of D and taking $t \rightarrow \infty$, we get $D(\nabla)(v) = 0$ since $\left(\frac{1}{t}\right)^{q-p} = t^{p-q} \rightarrow 0$ for $p < q$. \square

3. The main results. As the summary of Lemmas 1–3 we get the following main theorem.

Theorem 1. *There exists the bijection between the set of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operators $D: Q_{proj-proj}^\tau \rightsquigarrow QT^*$ and the set of sequences $(\tilde{D}_i)_{i=1,2,3,4}$ consisting of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operators $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3: Q_{proj-proj}^\tau \rightsquigarrow (T^*, T^* \otimes T^* \otimes T)$ and $\tilde{D}_4: Q_{proj-proj}^\tau \rightsquigarrow (T^*, T^* \otimes T^* \otimes T^*)$.*

More precisely, the system of operators $(\tilde{D}_i)_{i=1,2,3,4}$ defines a new sequence of operators $(D_i)_{i=1,\dots,8}$ (of the type from Lemma 1) such as

$$\begin{aligned} D_1 &:= \tilde{D}_1, & D_4 &:= \tilde{D}_2, & D_6 &:= \tilde{D}_3 \text{ (modulo the bijection from Lemma 2)} \\ D_2 &:= \tilde{D}_4, & D_3 &= 0, & D_5 &= 0, & D_7 &= 0, & D_8 &= 0. \end{aligned}$$

This system of operators $(D_i)_{i=1,\dots,8}$ defines the $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operator D (by Lemma 1).

Lemma 3 shows that the above assignment $(\tilde{D}_i)_{i=1,2,3,4} \mapsto D$ is a bijection.

Theorem 1 reduces the classification of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operators $D: Q_{proj-proj}^\tau \rightsquigarrow QT^*$ to the classification of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operators $D: Q_{proj-proj}^\tau \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$ for $p = 2, q = 1$ and $p = 3, q = 0$.

Definition 3. An $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operator

$$D: Q_{proj-proj}^\tau \rightsquigarrow \otimes^p T^* \otimes \otimes^q T$$

is an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -invariant family of regular operators

$$D = (D_Y): Q_{proj-proj}^\tau(Y) \rightarrow C_Y^\infty(\otimes^p T^* Y \otimes \otimes^q TY)$$

defined for every $Y \in Obj(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$, where $Q_{proj-proj}^\tau(Y)$ is defined in Definition 1 and $C_Y^\infty(\otimes^p T^* Y \otimes \otimes^q TY)$ means the set of smooth tensor fields on Y .

The $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -invariance of D means almost the same as in Definition 1, i.e. φ -related connections are transformed into φ -related tensor fields.

Example 4. An example of an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operator

$$D: Q_{proj-proj}^\tau \rightsquigarrow \otimes^3 T^* \otimes T$$

is a family $D = (R_Y)$ of operators

$$R_Y: Q_{proj-proj}^\tau(Y) \rightarrow C_Y^\infty(\otimes^3 T^* Y \otimes TY)$$

for any $Y \in Obj(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$, where $R_Y(\nabla)$ is the curvature tensor of ∇ .

Theorem 2. Let $p \geq q, r := p - q$. There exists the bijection between the set of all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operators $D: Q_{proj-proj}^\tau \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$ and the set of $(r + 1)$ -elements sequences $(D_i)_{i=0,1,\dots,r}$ consisting of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operators $D_i: Q_{proj-proj}^\tau \rightsquigarrow \otimes^p T^* \otimes \otimes^q T \otimes S^i T$, i.e. $D_i: Q_{proj-proj}^\tau \rightsquigarrow \otimes^p T^* \otimes \otimes^{q+i} T$ and $D_i(\nabla)(w_1, \dots, w_p, v_1, \dots, v_{q+i})$ is symmetric with respect to v_{q+1}, \dots, v_{q+i} .

Schema of the proof. Consider any $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operator

$$D: Q_{proj-proj}^\tau \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T).$$

Let $\nabla \in Q_{proj-proj}^\tau(\mathbb{R}^{m_1,m_2,n_1,n_2})$ and $v \in T_{(0,0,0,0)}^* \mathbb{R}^{m_1,m_2,n_1,n_2}$. We are going to study $D(\nabla)(v)$.

By the non-linear Petree theorem (see [3]) we have

$$D(\nabla)(v) = D(\tilde{\nabla})(v),$$

where $\tilde{\nabla}$ is some projectable-projectable classical torsion-free linear connection on $\mathbb{R}^{m_1, m_2, n_1, n_2}$ with Christoffel symbols $\tilde{\nabla}_{bc}^a$ being polynomials of degree k . Thus we have

$$\tilde{\nabla}_{bc}^a = \sum_{|\alpha| \leq k} \nabla_{bc; \alpha}^a x^\alpha,$$

where $\nabla_{bc; \alpha}^a \in \mathbb{R}$ and $x^1, \dots, x^{m_1+m_2+n_1+n_2}$ is the usual fibred-fibred coordinate system on $\mathbb{R}^{m_1, m_2, n_1, n_2}$.

In short, we write $D(\nabla)(v) = D(\nabla_{bc; \alpha}^a)(v)$.

Using the invariance of D with respect to homotheties $\frac{1}{t}id$, $t \neq 0$, we get the homogeneity condition

$$t^r D(\nabla_{bc; \alpha}^a)(v) = D(t^{|\alpha|+1} \nabla_{bc; \alpha}^a)(tv).$$

By the homogeneous function theorem (see [3]) and by the invariance of D with respect to $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -charts we get that $D(\nabla)(v)$ is a polynomial of degree not higher than $r := p - q$ with respect to $v \in T_y^* Y$, $y \in Y$, for every $Y \in \text{Obj}(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2})$ and $\nabla \in Q_{proj-proj}^r(Y)$.

Thus we have

$$D(\nabla)(tv) = \sum_{i=0}^r D_i(\nabla)(v) t^i$$

for some uniquely determined coefficients $D_i(\nabla)(v) \in \otimes^p T_y^* Y \otimes \otimes^q T_y Y$.

For every $a \in \mathbb{R}$ we have

$$D(\nabla)(tav) = \sum_{i=0}^r D_i(\nabla)(av) t^i$$

and

$$D(\nabla)(tav) = \sum_{i=0}^r D_i(\nabla)(v) a^i t^i,$$

hence we get

$$D(\nabla)(av) = a^i D_i(\nabla)(v).$$

It means that $D_i(\nabla)(v)$ is a polynomial of degree i with respect to v and it can be identified with the corresponding element

$$D_i(\nabla)(v) \in \otimes^p T_y^* Y \otimes \otimes^q T_y Y \otimes S^i T_y Y.$$

Summarizing, for every $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator

$$D: Q_{proj-proj}^r \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$$

we defined the sequence $(D_i)_{i=0,1,\dots,r}$ consisting of $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators

$$D_i: Q_{proj-proj}^r \rightsquigarrow \otimes^p T^* \otimes \otimes^q T \otimes S^i T.$$

Conversely, analysing the above reasoning, one can see that every $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator $D: Q_{proj-proj}^r \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$ can be reconstructed from the sequence $(D_i)_{i=0,1,\dots,r}$ of operators

$$D_i: Q_{proj-proj}^r \rightsquigarrow \otimes^p T^* \otimes \otimes^q T \otimes S^i T. \quad \square$$

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Anna Bednarska
 Institute of Mathematics
 Maria Curie-Skłodowska University
 pl. M. Curie-Skłodowskiej 1
 20-031 Lublin
 Poland
 e-mail: bednarska@hektor.umcs.lublin.pl

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