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Spacelike intersection curve of three spacelike hypersurfaces in E_1^4

Abstract. In this paper, we compute the Frenet vectors and the curvatures of the spacelike intersection curve of three spacelike hypersurfaces given by their parametric equations in four-dimensional Minkowski space E_1^4.

1. Introduction. The surface-surface intersection(SSI) is one of the basic problems in computational geometry. The main purpose here is to determine the intersection curve between the surfaces and to get information about the geometrical properties of the curve. Since the surfaces are mostly given by their parametric or implicit equations, three cases are valid for the SSI problems: *parametric-parametric*, *implicit-implicit* and *parametric-implicit*.

There are two types of SSI problems: *transversal* or *tangential*. The intersection at the intersecting points is called transversal if the normal vectors of the surfaces are linearly independent, and is called tangential if the normal vectors of the surfaces are linearly dependent. The tangent vector of the intersection curve can be obtained easily by the vector product of the normal vectors of the surfaces in transversal intersection problems. Therefore, so many studies have recently been done about this type of problems. Hartmann [6], provides formulas for computing the curvatures of the intersection curves for all types of intersection problems in three-dimensional Euclidean space. Willmore [11], and using the implicit function theorem

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Alessio [1], give the methods to compute the unit tangent, the unit principal normal, the unit binormal vectors, and then the curvature and the torsion of the transversal intersection curve of two implicit surfaces. Goldman [5], provides formulas for computing the curvature and the torsion of intersection curve of two implicit surfaces, using the classical curvature formulas in Differential Geometry. Ye and Maekawa [12], present algorithms to compute the Frenet vectors and curvatures of the intersection curve for all three types of transversal and tangential intersections. Alessio and Guadalupe [2], give formulas for computing the local properties of a transversal intersection curve of two spacelike surfaces in the Lorentz–Minkowski 3-space. Using the implicit function theorem, Alessio [3], presents algorithms for computing the differential geometry properties of intersection curves of three implicit surfaces in R^4. Düldül [4], gives methods for computing the Frenet apparatus of the transversal intersection curve of three parametric hypersurfaces in four-dimensional Euclidean space.

In this paper, we find the tangent, the principal normal, the first and second binormal vectors and the first, second and third curvatures of the spacelike intersection curve of three parametric spacelike hypersurfaces which are intersecting transversally in E_1^4.

2. Preliminaries. The Minkowski space E_1^4 is the real vector space R^4 provided with the (standard flat) metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of E_1^4. A vector $\mathbf{u} \in E_1^4$ is called a spacelike, a timelike, and a null (lightlike) vector if $g(\mathbf{u}, \mathbf{u}) > 0$ or $\mathbf{u} = 0$, $g(\mathbf{u}, \mathbf{u}) < 0$, and $g(\mathbf{u}, \mathbf{u}) = 0$ for $\mathbf{u} \neq 0$, respectively, [8]. The norm of a vector $\mathbf{u}$ is defined by $||\mathbf{u}|| = \sqrt{|g(\mathbf{u}, \mathbf{u})|}$. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be orthogonal if $g(\mathbf{u}, \mathbf{v}) = 0$. A vector $\mathbf{u}$ satisfying $g(\mathbf{u}, \mathbf{u}) = \pm 1$ is called a unit vector. For an arbitrary curve $\alpha = \alpha(s)$ in E_1^4, if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null vectors, then the curve is called a spacelike, a timelike or a null curve, respectively, [8].

A hypersurface in E_1^4 is called a timelike (spacelike) hypersurface if the induced metric on the hypersurface is a Lorentz (positive definite Riemannian) metric. The normal vector on the timelike (spacelike) hypersurface is a spacelike (timelike) vector.

The ternary product of the vectors $\mathbf{u} = \sum_{i=1}^4 u_i \mathbf{e}_i$, $\mathbf{v} = \sum_{i=1}^4 v_i \mathbf{e}_i$, and $\mathbf{w} = \sum_{i=1}^4 w_i \mathbf{e}_i$ is defined by

$$\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} = - \begin{vmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is the standard basis of four-dimensional Minkowski space E_1^4.

For the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ the following equations are satisfied: $\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_3 = \mathbf{e}_4$, $\mathbf{e}_2 \otimes \mathbf{e}_3 \otimes \mathbf{e}_4 = \mathbf{e}_1$, $\mathbf{e}_3 \otimes \mathbf{e}_4 \otimes \mathbf{e}_1 = \mathbf{e}_2$, $\mathbf{e}_4 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 = -\mathbf{e}_3$, [13].

From the definition of ternary product, we get $g(\mathbf{u}, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}) = g(\mathbf{v}, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}) = g(\mathbf{w}, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}) = 0$; that is, the vector $\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}$ is orthogonal to $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. The triple vector product of vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in E_1^3 is defined by

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{u} & \mathbf{v} \\ g(\mathbf{u}, \mathbf{w}) & g(\mathbf{v}, \mathbf{w}) \end{vmatrix} = g(\mathbf{v}, \mathbf{w})\mathbf{u} - g(\mathbf{u}, \mathbf{w})\mathbf{v},$$

a linear combination of vectors $\mathbf{u}$ and $\mathbf{v}$, [7, 10]. In an analogous manner in E_1^4 we can express the quintuple vector product of vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, and $\mathbf{y}$ as

$$(\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}) \otimes \mathbf{x} \otimes \mathbf{y} = \begin{vmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \\ g(\mathbf{u}, \mathbf{x}) & g(\mathbf{v}, \mathbf{x}) & g(\mathbf{w}, \mathbf{x}) \\ g(\mathbf{u}, \mathbf{y}) & g(\mathbf{v}, \mathbf{y}) & g(\mathbf{w}, \mathbf{y}) \end{vmatrix},$$

a linear combination of vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ (see [10] for the Euclidean case). We may also write

$$g(\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}, \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}) = - \begin{vmatrix} g(\mathbf{u}, \mathbf{x}) & g(\mathbf{v}, \mathbf{x}) & g(\mathbf{w}, \mathbf{x}) \\ g(\mathbf{u}, \mathbf{y}) & g(\mathbf{v}, \mathbf{y}) & g(\mathbf{w}, \mathbf{y}) \\ g(\mathbf{u}, \mathbf{z}) & g(\mathbf{v}, \mathbf{z}) & g(\mathbf{w}, \mathbf{z}) \end{vmatrix}.$$

Let $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}_1(s), \mathbf{b}_2(s)\}$ be the moving Frenet frame along the curve $\alpha(s)$ in the Minkowski 4-space E_1^4. Then $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}_1(s), \mathbf{b}_2(s)$ denote the tangent, the principal normal, the first binormal, and the second binormal vector fields, respectively.

Let $\alpha(s)$ be a spacelike curve with arc length parameter s in E_1^4. Then $\mathbf{t}(s) = \alpha'(s)$ is a spacelike unit vector, i.e., $||\alpha'(s)|| = 1$. Therefore $g(\alpha'(s), \alpha'(s)) = 1$ and $g(\alpha'(s), \alpha''(s)) = 0$. Depending on the vector $\alpha''(s)$ we investigate the following cases [9].

Case 1: $\mathbf{n}$ is spacelike:

Case 1.1: The second binormal vector $\mathbf{b}_2$ is the unique timelike vector in the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$. Then the Frenet formulas are

$$\begin{cases} \mathbf{t}' = k_1 \mathbf{n} \\ \mathbf{n}' = -k_1 \mathbf{t} + k_2 \mathbf{b_1} \\ \mathbf{b}_1' = -k_2 \mathbf{n} + k_3 \mathbf{b_2} \\ \mathbf{b}_2' = k_3 \mathbf{b_1}, \end{cases}$$

where k_i's $(i = 1, 2, 3)$ are the *ith* curvature functions of the curve α.

Case 1.2: The first binormal vector $\mathbf{b}_1$ is the unique timelike vector in the tetrad $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$. In this case, the Frenet formulas are given by

$$\begin{cases} \mathbf{t}' = k_1\mathbf{n} \\ \mathbf{n}' = -k_1\mathbf{t} + k_2\mathbf{b_1} \\ \mathbf{b}_1' = k_2\mathbf{n} + k_3\mathbf{b_2} \\ \mathbf{b}_2' = -k_3\mathbf{b_1}. \end{cases}$$

Case 2: $\mathbf{n}$ is timelike:
The Frenet formulas have the form

$$\begin{cases} \mathbf{t}' = k_1\mathbf{n} \\ \mathbf{n}' = k_1\mathbf{t} + k_2\mathbf{b_1} \\ \mathbf{b}_1' = k_2\mathbf{n} + k_3\mathbf{b_2} \\ \mathbf{b}_2' = -k_3\mathbf{b_1}. \end{cases}$$

From elementary differential geometry, we know that $\alpha'(s) = \mathbf{t}(s)$. Also, the second derivative $\alpha''(s)$ is equal to $k_1\mathbf{n}$ in all above cases. The third derivative $\alpha'''(s)$ can be obtained as $\alpha''' = k_1'\mathbf{n} + k_1\mathbf{n}'$ by differentiating the equation $\alpha'' = k_1\mathbf{n}$. If we replace $\mathbf{n}'$ using the Frenet formulas, we get

Case 1:

$$\alpha''' = -k_1^2\mathbf{t} + k_1'\mathbf{n} + k_1k_2\mathbf{b}_1. \tag{2.1}$$

Case 2:

$$\alpha''' = k_1^2\mathbf{t} + k_1'\mathbf{n} + k_1k_2\mathbf{b}_1. \tag{2.2}$$

Then the second curvature k_2 of the curve α can be obtained from Eq. (2.1) and Eq. (2.2) as

Case 1.1 and Case 2: $k_2 = \frac{g(\alpha''',\mathbf{b}_1)}{k_1}$.

Case 1.2: $k_2 = -\frac{g(\alpha''',\mathbf{b}_1)}{k_1}$.

Now let us find the fourth derivative $\alpha^{(4)}(s)$ similar to the third derivative $\alpha'''(s)$:

$$\begin{aligned} \text{Case 1.1: } \alpha^{(4)} &= -3k_1k_1'\mathbf{t} + (-k_1^3 + k_1'' - k_1k_2^2)\mathbf{n} + (2k_1'k_2 + k_1k_2')\mathbf{b}_1 \\ &+ k_1k_2k_3\mathbf{b}_2 \\ \text{Case 1.2: } \alpha^{(4)} &= -3k_1k_1'\mathbf{t} + (-k_1^3 + k_1'' + k_1k_2^2)\mathbf{n} + (2k_1'k_2 + k_1k_2')\mathbf{b}_1 \\ &+ k_1k_2k_3\mathbf{b}_2 \end{aligned}$$

$$\text{Case 2: } \alpha^{(4)} = 3k_1k_1'\mathbf{t} + (k_1^3 + k_1'' + k_1k_2^2)\mathbf{n} + (2k_1'k_2 + k_1k_2')\mathbf{b}_1 + k_1k_2k_3\mathbf{b}_2.$$

Using the above equations, the third curvature of α can be found by $k_3 = -\frac{g(\alpha^{(4)},\mathbf{b}_2)}{k_1k_2}$ for Case 1.1 and $k_3 = \frac{g(\alpha^{(4)},\mathbf{b}_2)}{k_1k_2}$ for Case 1.2 and Case 2.

3. The curvatures of spacelike intersection curve. Let M_i $(i = 1, 2, 3)$ be three spacelike hypersurfaces with parametric equations $X^i = X^i(u_i, v_i, w_i)$. Let us assume these hypersurfaces intersect transversally at an intersection point $\alpha(s_0) = P$ on the spacelike intersection curve $\alpha(s)$.

Then, the unit normal vectors $\mathbf{N}_i$ of these hypersurfaces are timelike vectors and they can be given by

$$\mathbf{N}_i = \frac{X^i_{u_i} \otimes X^i_{v_i} \otimes X^i_{w_i}}{||X^i_{u_i} \otimes X^i_{v_i} \otimes X^i_{w_i}||}.$$

Since the intersection curve α is a spacelike curve, the unit tangent vector $\mathbf{t}$ of α is a spacelike vector and can be obtained by the ternary product of the normal vectors at P:

$$\mathbf{t} = \frac{\mathbf{N}_1 \otimes \mathbf{N}_2 \otimes \mathbf{N}_3}{||\mathbf{N}_1 \otimes \mathbf{N}_2 \otimes \mathbf{N}_3||}.$$

Since the intersection is transversal, the normal vectors $\mathbf{N}_i$ are linearly independent at the intersection point, i.e., $\mathbf{N}_1 \otimes \mathbf{N}_2 \otimes \mathbf{N}_3 \neq 0$.

3.1. First Curvature. From the Frenet equations, we know $k_1 = ||\mathbf{t}'||$. Then we must find the vector $\mathbf{t}'$ for calculating the first curvature k_1. Since $\mathbf{t}'$ is orthogonal to $\mathbf{t}$, we can write

$$\alpha'' = \mathbf{t}' = a_1\mathbf{N}_1 + a_2\mathbf{N}_2 + a_3\mathbf{N}_3, \; a_i \in \mathbb{R}. \tag{3.1}$$

Now let us determine the scalars a_i to find α''. If we take the dot product of both hand sides of (3.1) with $\mathbf{N}_i$, then we get

$$g(\mathbf{N}_1, \mathbf{N_i})a_1 + g(\mathbf{N}_2, \mathbf{N_i})a_2 + g(\mathbf{N}_3, \mathbf{N_i})a_3 = K^i_n, \tag{3.2}$$

where $K^i_n = g(\mathbf{t}', \mathbf{N}_i)$, $(i = 1, 2, 3)$. The coefficients determinant of the linear system (3.2) is $\Delta = -||\mathbf{N}_1 \otimes \mathbf{N}_2 \otimes \mathbf{N}_3||^2$. Since Δ is different from zero, solving the coefficients from linear system (3.2) yields (as in [4])

$$a_i = \frac{1}{\Delta}\{-\sinh^2\theta_{jk}K^i_n + b_{ij}K^j_n + b_{ik}K^k_n\}, \quad i, j, k = 1, 2, 3 \text{ (cyclic)},$$

where θ_{ij} is the angle between the timelike unit normal vectors $\mathbf{N}_i$ and $\mathbf{N}_j$. Also, if $\mathbf{N}_1, \mathbf{N}_2$ and $\mathbf{N}_3$ are in the same timecone of E^4_1, then

$$b_{ij} = \cosh\theta_{ik}\cosh\theta_{jk} - \cosh\theta_{ij}.$$

If $\mathbf{N}_1, \mathbf{N}_2$ and $\mathbf{N}_3$ are not in the same timecone of E^4_1, then

$$b_{ij} = \cosh\theta_{ik}\cosh\theta_{jk} + \cosh\theta_{ij}.$$

If $\mathbf{N}_1$ and $\mathbf{N}_2$ are in the same timecone but $\mathbf{N}_3$ is in the different timecone, then

$$b_{ij} = (-1)^{i+k}\cosh\theta_{ik}\cosh\theta_{jk} - \cosh\theta_{ij}.$$

If $\mathbf{N}_1$ and $\mathbf{N}_3$ are in the same timecone but $\mathbf{N}_2$ is in the different timecone, then

$$b_{ij} = (-1)^{i+j}\cosh\theta_{ik}\cosh\theta_{jk} - \cosh\theta_{ij}.$$

If $\mathbf{N}_2$ and $\mathbf{N}_3$ are in the same timecone but $\mathbf{N}_1$ is in the different timecone, then

$$b_{ij} = (-1)^{j+k}\cosh\theta_{ik}\cosh\theta_{jk} - \cosh\theta_{ij}.$$

The normal curvatures K_n^i needed to find the scalars a_i are calculated as expressed in [4]. Then the first curvature k_1 of the intersection curve α at P is given as follows:
If $\mathbf{N}_1, \mathbf{N}_2$ and $\mathbf{N}_3$ are in the same timecone, then

$$k_1^2 = \left| a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 \cosh\theta_{12} + 2a_1a_3 \cosh\theta_{13} + 2a_2a_3 \cosh\theta_{23} \right|.$$

If $\mathbf{N}_1, \mathbf{N}_2$ and $\mathbf{N}_3$ are not in the same timecone, then

$$k_1^2 = \left| a_1^2 + a_2^2 + a_3^2 - 2a_1a_2 \cosh\theta_{12} - 2a_1a_3 \cosh\theta_{13} - 2a_2a_3 \cosh\theta_{23} \right|.$$

If $\mathbf{N}_1$ and $\mathbf{N}_2$ are in the same timecone but $\mathbf{N}_3$ is in the different timecone, then

$$k_1^2 = \left| a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 \cosh\theta_{12} - 2a_1a_3 \cosh\theta_{13} - 2a_2a_3 \cosh\theta_{23} \right|.$$

If $\mathbf{N}_1$ and $\mathbf{N}_3$ are in the same timecone but $\mathbf{N}_2$ is in the different timecone, then

$$k_1^2 = \left| a_1^2 + a_2^2 + a_3^2 - 2a_1a_2 \cosh\theta_{12} + 2a_1a_3 \cosh\theta_{13} - 2a_2a_3 \cosh\theta_{23} \right|.$$

If $\mathbf{N}_2$ and $\mathbf{N}_3$ are in the same timecone but $\mathbf{N}_1$ is in the different timecone, then

$$k_1^2 = \left| a_1^2 + a_2^2 + a_3^2 - 2a_1a_2 \cosh\theta_{12} - 2a_1a_3 \cosh\theta_{13} + 2a_2a_3 \cosh\theta_{23} \right|.$$

3.2. Second Curvature. Now, let us find the second curvature of the intersection curve α at P. Since the timelike unit normal vectors $\mathbf{N}_i$ are orthogonal to $\mathbf{t}$, the terms $k_1'\mathbf{n} + k_1k_2\mathbf{b}_1$ in Eq. (2.1) and Eq. (2.2) can be replaced by $c_1\mathbf{N}_1 + c_2\mathbf{N}_2 + c_3\mathbf{N}_3$. Thus

Case 1: $\alpha''' = -k_1^2\mathbf{t} + c_1\mathbf{N}_1 + c_2\mathbf{N}_2 + c_3\mathbf{N}_3$,

Case 2: $\alpha''' = k_1^2\mathbf{t} + c_1\mathbf{N}_1 + c_2\mathbf{N}_2 + c_3\mathbf{N}_3$.

If the dot products of both hand sides of above equations are taken with $\mathbf{N}_i$, we have a linear equation system similar to (3.2). Solving this system yields

$$(3.3) \qquad c_i = \frac{1}{\Delta}\{-\sinh^2\theta_{jk}\mu_i + b_{ij}\mu_j + b_{ik}\mu_k\}, \quad i, j, k = 1, 2, 3 \text{ (cyclic)},$$

where $\mu_i = g(\alpha''', \mathbf{N}_i)$. The scalars μ_i in Eq. (3.3) are computed as explained in [4]. Then, the second curvature k_2 of the intersection curve α at P is given by

Case 1.1 and Case 2: $k_2 = \frac{g(\alpha''', \mathbf{b}_1)}{k_1}$,

Case 1.2: $k_2 = -\frac{g(\alpha''', \mathbf{b}_1)}{k_1}$.

3.3. Third Curvature. Now, let us compute the third curvature k_3 of the curve α at P. Similar to the second and third derivatives of α, we may write

Case 1: $\alpha^{(4)} = -3k_1k_1'\mathbf{t} + d_1\mathbf{N}_1 + d_2\mathbf{N}_2 + d_3\mathbf{N}_3$, Case 2: $\alpha^{(4)} = 3k_1 k_1' \mathbf{t} + d_1 \mathbf{N}_1 + d_2 \mathbf{N}_2 + d_3 \mathbf{N}_3$, where

$$d_i = \frac{1}{\Delta}\{-\sinh^2\theta_{jk}\xi_i + b_{ij}\xi_j + b_{ik}\xi_k\}, \quad i,j,k = 1,2,3 \text{ (cyclic)},$$

and $\xi_i = g(\alpha^{(4)}, \mathbf{N}_i)$. Obtaining the scalars ξ_i is mentioned in [4].

Thus, the third curvature of the intersection curve is found as

Case 1.1: $k_3 = -\frac{g(\alpha^{(4)}, \mathbf{b}_2)}{k_1 k_2}$,

Case 1.2 and Case 2: $k_3 = \frac{g(\alpha^{(4)}, \mathbf{b}_2)}{k_1 k_2}$.

4. Examples.

4.1. Example 1. Let us consider the spacelike hypersurfaces

$$M_1 : X^1(u_1, v_1, w_1) = \left(\cosh u_1 + \frac{1}{2}, \sinh u_1, v_1, w_1\right),$$

$$M_2 : X^2(u_2, v_2, w_2) = (\cosh u_2, \sinh u_2 \cos v_2, \sinh u_2 \sin v_2, w_2),$$

$$M_3 : X^3(u_3, v_3, w_3)$$

$$= \left(\frac{1}{\sqrt{2}}\cosh u_3 + \frac{1}{2}, \frac{1}{\sqrt{2}}\sinh u_3 \sin v_3 + \frac{1}{2}, w_3, \frac{1}{\sqrt{2}}\sinh u_3 \cos v_3\right)$$

in the Minkowski 4-space, where $u_2 \neq 0$, $u_3 \neq 0$. Let us compute the Frenet apparatus of the spacelike intersection curve α at the intersection point

$$P = X^1\left(0, \frac{\sqrt{5}}{2}, \frac{1}{2}\right) = X^2\left(\ln\left(\frac{3+\sqrt{5}}{2}\right), \frac{\pi}{2}, \frac{1}{2}\right)$$

$$= X^3\left(\ln\left(\sqrt{2}+1\right), \frac{7\pi}{4}, \frac{\sqrt{5}}{2}\right) = \left(\frac{3}{2}, 0, \frac{\sqrt{5}}{2}, \frac{1}{2}\right).$$

At this point, the timelike unit normal vectors of these hypersurfaces are, respectively,

$$\mathbf{N}_1 = (1,0,0,0),\ \mathbf{N}_2 = \left(\frac{3}{2}, 0, \frac{\sqrt{5}}{2}, 0\right),\ \mathbf{N}_3 = \left(\sqrt{2}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

and the spacelike unit tangent vector of α is

$$\mathbf{t} = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right).$$

Since $g(\mathbf{N}_1, \mathbf{N}_2)$, $g(\mathbf{N}_1, \mathbf{N}_3)$ and $g(\mathbf{N}_2, \mathbf{N}_3)$ are smaller than zero, the timelike unit normal vectors $\mathbf{N}_i$, $1 \leq i \leq 3$, are in the same timecone. Then, $\cosh\theta_{12} = \frac{3}{2}$, $\cosh\theta_{13} = \sqrt{2}$, $\cosh\theta_{23} = \frac{3}{\sqrt{2}}$ and so $b_{12} = \frac{3}{2}$, $b_{13} = \frac{5}{2\sqrt{2}}$ and $b_{23} = 0$.

The non-vanishing first fundamental form coefficients of these hypersurfaces at P are

$$g^1_{11} = g^1_{22} = g^1_{33} = g^2_{11} = g^2_{33} = g^3_{33} = 1,\ g^2_{22} = \frac{5}{4},\ g^3_{11} = g^3_{22} = \frac{1}{2}$$

and the non-vanishing second fundamental form coefficients of these hypersurfaces at P are

$$h^1_{11} = h^2_{11} = -1,\ h^2_{22} = -\frac{5}{4},\ h^3_{11} = h^3_{22} = -\frac{1}{\sqrt{2}}.$$

As expressed in [4], the values u'_i, v'_i and w'_i are found as:

$$\begin{aligned} u'_1 &= \frac{1}{\sqrt{2}}, & v'_1 &= 0, & w'_1 &= \frac{1}{\sqrt{2}}, \\ u'_2 &= 0, & v'_2 &= -\frac{\sqrt{2}}{\sqrt{5}}, & w'_2 &= \frac{1}{\sqrt{2}}, \\ u'_3 &= 0, & v'_3 &= \sqrt{2}, & w'_3 &= 0. \end{aligned}$$

Hence, for the normal curvatures K^i_n we find $K^1_n = K^2_n = -\frac{1}{2}$, $K^3_n = -\sqrt{2}$. Thus, we have

$$\begin{aligned} &a_1 = \frac{6}{5}, \quad a_2 = \frac{1}{5}, \quad a_3 = -\frac{1}{\sqrt{2}}, \quad k_1 = \frac{\sqrt{3}}{\sqrt{10}}, \\ &\alpha'' = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2\sqrt{5}}, -\frac{1}{2}\right), \quad \mathbf{n} = \left(\frac{\sqrt{5}}{\sqrt{6}}, \frac{\sqrt{5}}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{\sqrt{5}}{\sqrt{6}}\right), \end{aligned}$$

where $\mathbf{n}$ is a spacelike vector. Also, the values u''_i, v''_i and w''_i are calculated as:

$$\begin{aligned} u''_1 &= \frac{1}{2}, & v''_1 &= \frac{1}{2\sqrt{5}}, & w''_1 &= -\frac{1}{2}, \\ u''_2 &= \frac{1}{\sqrt{5}}, & v''_2 &= -\frac{1}{\sqrt{5}}, & w''_2 &= -\frac{1}{2}, \\ u''_3 &= \frac{1}{\sqrt{2}}, & v''_3 &= 0, & w''_3 &= \frac{1}{2\sqrt{5}}. \end{aligned}$$

Using above, we get $\mu_1 = \mu_2 = -\frac{3}{2\sqrt{2}}$ and $\mu_3 = 0$. So, the coefficients c_i are found as $c_1 = -\frac{12}{5\sqrt{2}}$, $c_2 = \frac{3}{5\sqrt{2}}$, $c_3 = \frac{3}{2}$. Then, we get

$$\begin{aligned} &\alpha''' = \left(\frac{3}{2\sqrt{2}}, -\frac{9}{5\sqrt{2}}, \frac{3\sqrt{5}}{10\sqrt{2}}, \frac{6}{5\sqrt{2}}\right), \\ &\alpha' \otimes \alpha'' \otimes \alpha''' = \left(\frac{3}{2\sqrt{5}}, 0, \frac{3}{2}, 0\right), \quad ||\alpha' \otimes \alpha'' \otimes \alpha'''|| = \frac{3}{\sqrt{5}}, \end{aligned}$$

$$\mathbf{b}_2 = \frac{\alpha' \otimes \alpha'' \otimes \alpha'''}{||\alpha' \otimes \alpha'' \otimes \alpha'''||} = \left(\frac{1}{2}, 0, \frac{\sqrt{5}}{2}, 0\right),$$
$$\mathbf{b}_1 = \frac{\mathbf{b}_2 \otimes \alpha' \otimes \alpha''}{||\mathbf{b}_2 \otimes \alpha' \otimes \alpha''||} = \left(\frac{5}{2\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{\sqrt{5}}{2\sqrt{3}}, -\frac{1}{\sqrt{3}}\right).$$

Note that $\mathbf{b}_2$ is a spacelike vector and $\mathbf{b}_1$ is the unique timelike vector in the tetrad $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$. Because of this, Case 1.2 is valid. Then the second curvature k_2 of α is found as $k_2 = 2\sqrt{5}$. Also, $k_1' = g(\alpha''', \mathbf{n}) = -\frac{7\sqrt{15}}{10}$.

For the values u_i''', v_i''' and w_i''', we obtain

$$u_1''' = -\frac{23}{10\sqrt{2}}, \quad v_1''' = \frac{3\sqrt{5}}{10\sqrt{2}}, \quad w_1''' = \frac{6}{5\sqrt{2}},$$
$$u_2''' = \frac{3}{\sqrt{10}}, \quad v_2''' = \frac{32}{5\sqrt{10}}, \quad w_2''' = \frac{6}{5\sqrt{2}},$$
$$u_3''' = \frac{3}{2}, \quad v_3''' = -\frac{13}{5\sqrt{2}}, \quad w_3''' = \frac{3}{2\sqrt{10}}.$$

Hence, we have

$$\xi_1 = \frac{18}{5}, \quad \xi_2 = \frac{69}{20}, \quad \xi_3 = -\frac{21}{10\sqrt{2}}$$

and using these values

$$d_1 = \frac{201}{25}, \quad d_2 = -\frac{39}{25}, \quad d_3 = -\frac{93}{10\sqrt{2}}$$

are found. So, we obtain

$$\alpha^{(4)} = \left(-\frac{18}{5}, \frac{38}{5}, -\frac{39}{10\sqrt{5}}, -\frac{3}{2}\right),$$
$$k_3 = -\frac{3}{20\sqrt{6}}.$$

4.2. Example 2. Let M_1, M_2, and M_3 be the spacelike hypersurfaces given by, respectively,

$$X^1(u_1, v_1, w_1) = \left(\cosh u_1, \sinh u_1 + \frac{1}{2}, v_1, w_1\right),$$
$$X^2(u_2, v_2, w_2) = \left(\frac{1}{2}\cosh u_2, \frac{1}{2}\sinh u_2 \cos v_2, w_2, \frac{1}{2}\sinh u_2 \sin v_2\right),$$
$$X^3(u_3, v_3, w_3) = \left(\frac{1}{2}\cosh u_3, w_3, \frac{1}{2}\sinh u_3 \sin v_3 - \frac{1}{2}, \frac{1}{2}\sinh u_3 \cos v_3\right),$$

where $u_2 \neq 0$, $u_3 \neq 0$. The Frenet vectors and the curvatures of the spacelike intersection curve of these surfaces at the intersection point

$$P = X^1\left(0, 0, \frac{1}{\sqrt{2}}\right) = X^2\left(\ln\left(2+\sqrt{3}\right), \arctan\sqrt{2}, 0\right)$$
$$= X^3\left(\ln\left(2+\sqrt{3}\right), \arctan\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \left(1, \frac{1}{2}, 0, \frac{1}{\sqrt{2}}\right)$$

are found as:

$$\mathbf{t} = \left(0, -\frac{\sqrt{2}}{\sqrt{5}}, -\frac{\sqrt{2}}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \quad \mathbf{n} = \left(\frac{5}{\sqrt{15}}, -\frac{1}{\sqrt{15}}, -\frac{1}{\sqrt{15}}, -\frac{2\sqrt{2}}{\sqrt{15}}\right),$$
$$\mathbf{b}_1 = \left(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{3}}\right), \quad \mathbf{b}_2 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right),$$
$$k_1 = \frac{2\sqrt{15}}{25}, \quad k_2 = 2, \quad k_3 = 0.$$

Here, the timelike unit normal vectors $\mathbf{N}_2$ and $\mathbf{N}_3$ are in the same timecone but $\mathbf{N}_1$ is in the different timecone. Also, $\mathbf{n}$ is the unique timelike vector in the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ and for this reason, Case 2 is valid.

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