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Estimates of L_p norms for sums of positive functions

ABSTRACT. We present new inequalities of L_p norms for sums of positive functions. These inequalities are useful for investigation of convergence of simple partial fractions in $L_p(\mathbb{R})$.

Let p_n be a polynomial of degree n with zeros $z_1, z_2, \ldots, z_n$. The logarithmic derivative of p_n

$$g_n(t) = \frac{p_n'(t)}{p_n(t)} = \sum_{k=1}^{n} \frac{1}{t - z_k}$$

is called a simple partial fraction.

Let $z_k = x_k + iy_k$. V. Yu. Protasov [1] showed that if

$$\sum_{k=1}^{\infty} \frac{1}{|y_k|^{1/q}} < +\infty, \ \frac{1}{p} + \frac{1}{q} = 1, \tag{1}$$

then the series

$$g_\infty(t) = \sum_{k=1}^{\infty} \frac{1}{t - z_k}$$

converges in $L_p(\mathbb{R})$.

In [1] the problem to find necessary and sufficient conditions for convergence of the series g_∞ in $L_p(\mathbb{R})$ was posed. Protasov proved that if g_∞

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converges in $L_p(\mathbb{R})$ and all z_k lie in the angle $|z| \le C|y|$ with a fixed C, then for all $\varepsilon > 0$ the following condition holds:

$$\sum_{k=1}^{\infty} \frac{1}{|y_k|^{1/q+\varepsilon}} < +\infty \tag{2}$$

Therefore, we see that the sufficient condition (1) is quite close to the necessary condition (2).

In the paper [2] we proved the following theorem.

Theorem 1. *Let $p > 1$. If*

$$\sum_{k=1}^{\infty} \frac{k^{p-1}}{|y_k|^{p-1}} < +\infty, \tag{3}$$

then the series

$$g_\infty(t) = \sum_{k=1}^{\infty} \frac{1}{t - z_k}$$

converges in $L_p(\mathbb{R})$. Conversely, if $g_\infty(t)$ converges in $L_p(\mathbb{R})$, the sequence $|y_n|$ is increasing and $|z_k| \le C|y_k|$, then the condition (3) *holds.*

The proof of Theorem 1 is based on the following fact.

For any $p \ge 2$ there exists a positive constant C_p depending only on p such that the following inequality holds

$$\int_{-\infty}^{+\infty} \left(\sum_{k=1}^{n} \frac{y_k}{(t - x_k)^2 + y_k^2} \right)^p dt \le C_p \sum_{k=1}^{n} \left| \frac{k}{y_k} \right|^{p-1}.$$

It turns out that there exists a nontrivial generalization of this result for arbitrary positive functions from arbitrary measurable space.

To be precise, let X be a measurable space with positive measure μ. Suppose that $f_k \in L_1(X,\mu) \cap L_\infty(X,\mu)$ and $f_k \ge 0$, $k = 1, 2, \dots, n$.

We set

$$L = \max_{1 \le k \le n} \int_X f_k d\mu,$$

$$M_k = ||f_k||_\infty.$$

The aim of the present paper is the following theorem.

Theorem 2. *If $p \in (1, 2]$, then there exists C_p such that*

$$\int_X \left(\sum_{k=1}^{n} f_k \right)^p d\mu \le C_p L \sum_{j=1}^{n} \left(\sum_{k=j}^{n} M_k \right)^{p-1}. \tag{4}$$

If $p \in [2, +\infty)$, then there exists C_p such that

$$\int_X \left(\sum_{k=1}^n f_k\right)^p d\mu \le C_p L \sum_{k=1}^n (kM_k)^{p-1}. \tag{5}$$

To prove Theorem 2 we need the following

Lemma. *For any natural p the following inequality holds*

$$\int_X \left(\sum_{k=1}^n f_k\right)^p d\mu \le p!(p-1)! L \sum_{k=1}^n (kM_k)^{p-1}. \tag{6}$$

Proof. We multiply out and then integrate term by term:

$$\begin{aligned}
&\int_X \left(\sum_{k=1}^n f_k\right)^p d\mu \\
&\quad = \sum_{k_1, k_2, \ldots, k_p} \int_X f_{k_1} f_{k_2} \cdots f_{k_p} d\mu \\
&\quad \le p! \sum_{k_1 \ge k_2 \ge \cdots \ge k_p} \int_X f_{k_1} f_{k_2} \cdots f_{k_p} d\mu \\
&\quad \le p! \sum_{k_1 \ge k_2 \ge \cdots \ge k_p} \int_X M_{k_1} M_{k_2} \cdots M_{k_{p-1}} f_{k_p} d\mu \\
&\quad = p! \sum_{k_1 \ge k_2 \ge \cdots \ge k_{p-1}} M_{k_1} M_{k_2} \cdots M_{k_{p-1}} \sum_{k_p=1}^{k_{p-1}} \int_X f_{k_p} d\mu \\
&\quad \le p! L \sum_{k_1 \ge k_2 \ge \cdots \ge k_{p-1}} M_{k_1} M_{k_2} \cdots M_{k_{p-2}} k_{p-1} M_{k_{p-1}}.
\end{aligned}$$

In these inequalities the indexes $k_1, k_2, \ldots, k_p$ are varying from 1 to n. We note that for $p = 1$ the last sum is equal to $n\pi$. For $p = 2$ that sum is equal to $2\pi \sum_{k=1}^n kM_k$.

It is clear that to prove (6) it is enough to show that

$$\sum_{k_1=1}^n M_{k_1} \sum_{k_2=1}^{k_1} M_{k_2} \cdots \sum_{k_{p-2}=1}^{k_{p-3}} M_{k_{p-2}} \sum_{k_{p-1}=1}^{k_{p-2}} M_{k_{p-1}} k_{p-1} \le (p-1)! \sum_{k=1}^n (kM_k)^{p-1}.$$

This inequality was established in the paper [2]. Lemma is proved. □

Proof of Theorem 2. We have

$$\int_X \left(\sum_{k=1}^n f_k\right)^p d\mu = \int_X \left(\sum_{k=1}^n f_k\right) \left(\sum_{k=1}^n f_k\right)^{p-1} d\mu \le 2^{p-1}(I_1 + I_2),$$

where

$$I_1 = \int\limits_X \sum_{j=1}^n f_j \left(\sum_{k=1}^j f_k \right)^{p-1} d\mu,$$

$$I_2 = \int\limits_X \sum_{j=1}^n f_j \left(\sum_{k=j+1}^n f_k \right)^{p-1} d\mu.$$

Here we have used the classical inequality $(a+b)^\alpha \le 2^\alpha(a^\alpha + b^\alpha)$ which holds for all positive a, b, α.

It is easy to see that

$$I_2 \le \int\limits_X \sum_{j=1}^n f_j \left(\sum_{k=j+1}^n M_k \right)^{p-1} d\mu \le L \sum_{j=1}^n \left(\sum_{k=j+1}^n M_k \right)^{p-1}. \tag{7}$$

Further we shall consider the cases $p \le 2$ and $p > 2$ separately.

Case $p \in (1, 2]$.

To get an upper estimate for I_1 we use the Hölder inequality

$$I_1 \le \sum_{j=1}^n \left(\int\limits_X f_j^\alpha d\mu \right)^{1/\alpha} \left(\int\limits_X \left(\sum_{k=1}^j f_k \right)^{(p-1)\beta} d\mu \right)^{1/\beta}$$

with parameters $\alpha = 1/(2-p)$, $\beta = 1/(p-1)$. Therefore,

$$\begin{aligned} I_1 &\le \sum_{j=1}^n \left(\int\limits_X f_j^\alpha d\mu \right)^{2-p} \left(\int\limits_X \sum_{k=1}^j f_k d\mu \right)^{p-1} \\ &= \sum_{j=1}^n \left(\int\limits_X f_j^{\alpha-1} f_j d\mu \right)^{2-p} \left(\int\limits_X \sum_{k=1}^j f_k d\mu \right)^{p-1} \\ &\le \sum_{j=1}^n \left(M_j^{\alpha-1} L \right)^{2-p} (jL)^{p-1} = L \sum_{j=1}^n (jM_j)^{p-1}. \end{aligned}$$

Applying Copson's inequality ([3], Theorem 344)

$$\sum_{n=1}^\infty (a_n + a_{n+1} + \cdots)^{p-1} > (p-1)^{p-1} \sum_{n=1}^\infty (na_n)^{p-1}$$

we get

$$I_1 \le L \sum_{j=1}^n \left(\sum_{k=j}^n M_k \right)^{p-1}.$$

This inequality together with (7) gives us desired estimate (4) which proves Theorem 2 in case when $p \le 2$.

Case $p \in (2, +\infty)$**.**

It follows from Copson's inequality ([3], Theorem 331)

$$\sum_{n=1}^{\infty}(a_n + a_{n+1} + \cdots)^{p-1} \leq (p-1)^{p-1}\sum_{n=1}^{\infty}(na_n)^{p-1}$$

that

$$I_2 \leq L(p-1)^{p-1}\sum_{j=1}^{n} j^{p-1}M_j^{p-1}. \tag{8}$$

To estimate I_1 we again use the Hölder inequality

$$I_1 \leq \sum_{j=1}^{n}\left(\int\limits_X f_j^{\alpha}d\mu\right)^{1/\alpha}\left(\int\limits_X \left(\sum_{k=1}^{j} f_k\right)^{(p-1)\beta} d\mu\right)^{1/\beta}$$

with parameters $\alpha = m/(m+1-p)$, $\beta = m/(p-1)$ where m is the integer part of p. Further, Lemma and Hölder's inequality yield the following estimates

$$\begin{aligned}
I_1 &\leq \sum_{j=1}^{n} L^{1/\alpha}M_j^{(\alpha-1)/\alpha}\left(\int\limits_X \left(\sum_{k=1}^{j} f_k\right)^{m} d\mu\right)^{(p-1)/m} \\
&\leq L\sum_{j=1}^{n} M_j^{(p-1)/m}\left(\pi m!(m-1)!\sum_{k=1}^{j}(kM_k)^{m-1}\right)^{(p-1)/m} \\
&= LC(m,p)\sum_{j=1}^{n}(jM_j)^{(p-1)/m}\left(\frac{1}{j}\sum_{k=1}^{j}(kM_k)^{m-1}\right)^{(p-1)/m} \\
&\leq LC(m,p)\left(\sum_{j=1}^{n}(jM_j)^{\alpha_1(p-1)/m}\right)^{1/\alpha_1} \\
&\qquad \times\left(\sum_{j=1}^{n}\left(\frac{1}{j}\sum_{k=1}^{j}(kM_k)^{m-1}\right)^{\beta_1(p-1)/m}\right)^{1/\beta_1}.
\end{aligned}$$

Setting $\alpha_1 = m$, $\beta_1 = m/(m-1)$, we obtain

$$\begin{aligned}
I_1 \leq LC(m,p)&\left(\sum_{j=1}^{n}(jM_j)^{p-1}\right)^{1/m} \\
&\times\left(\sum_{j=1}^{n}\left(\frac{1}{j}\sum_{k=1}^{j}(kM_k)^{m-1}\right)^{(p-1)/(m-1)}\right)^{(m-1)/m}.
\end{aligned}$$

Applying Hardy's inequality ([3], Theorem 326)

$$\sum_{n=1}^{\infty}\left(\frac{a_1+a_2+\cdots+a_k}{k}\right)^s \le \left(\frac{s}{s-1}\right)^s \sum_{k=1}^{\infty} a_k^s$$

with $s=(p-1)/(m-1)$ and $a_k=(kM_k)^{m-1}$ we see that

$$I_1 \le LC(m,p)\left(\sum_{j=1}^{n}(jM_j)^{p-1}\right)^{1/m}\left(s^s(s-1)^{-s}\sum_{j=1}^{n}(jM_j)^{p-1}\right)^{(m-1)/m}$$
$$= LC_1(m,p)\sum_{j=1}^{n}(jM_j)^{p-1}\,.$$

This estimate together with (8) proves (5). Theorem 2 is proved. □

Let us remark that the inequalities (4) and (5) are sharp up to some absolute constant depending on p only. This can be easily seen by setting $f_j\equiv 1$ on X. Other examples can be constructed as follows: $X=\mathbb{R}$ and

$$f_k(t)=\frac{y_k}{(t-x_k)^2+y_k^2}.$$

In case when $|y_k|$ is increasing sequence, it was proved in the paper [2] that the sign $\le$ in the inequalities (4) and (5) can replaced by $\ge$ with some other absolute constant $c_p>0$ depending on p only.

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