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On path-quasar Ramsey numbers

Abstract. Let G_1 and G_2 be two given graphs. The Ramsey number $R(G_1, G_2)$ is the least integer r such that for every graph G on r vertices, either G contains a G_1 or $\overline{G}$ contains a G_2. Parsons gave a recursive formula to determine the values of $R(P_n, K_{1,m})$, where P_n is a path on n vertices and $K_{1,m}$ is a star on $m+1$ vertices. In this note, we study the Ramsey numbers $R(P_n, K_1 \vee F_m)$, where F_m is a linear forest on m vertices. We determine the exact values of $R(P_n, K_1 \vee F_m)$ for the cases $m \leq n$ and $m \geq 2n$, and for the case that F_m has no odd component. Moreover, we give a lower bound and an upper bound for the case $n+1 \leq m \leq 2n-1$ and F_m has at least one odd component.

1. Introduction. We use Bondy and Murty [1] for terminology and notation not defined here, and consider finite simple graphs only.

Let G be a graph. We denote by $\nu(G)$ the order of G, by $\delta(G)$ the minimum degree of G, by $\omega(G)$ the number of components of G, and by $o(G)$ the number of components of G with an odd order.

Let G_1 and G_2 be two graphs. The *Ramsey number* $R(G_1, G_2)$, is defined as the least integer r such that for every graph G on r vertices, either G contains a G_1 or $\overline{G}$ contains a G_2, where $\overline{G}$ is the complement of G. If G_1 and G_2 are both complete, then $R(G_1, G_2)$ is the classical Ramsey number $r(\nu(G_1), \nu(G_2))$. Otherwise, $R(G_1, G_2)$ is usually called the *generalized*

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Ramsey number. We refer the reader to Graham et al. [5] for an introduction to the area of Ramsey theory.

We denote by P_n the path on n vertices. The graph $K_{1,m}$, $m \geq 2$, is called a *star*. The only vertex of degree m is called the *center* of the star. In 1974, Parsons [7] determined $R(P_n, K_{1,m})$ for all n, m. We list Parsons' result as below.

Theorem 1 (Parsons [7]).

$$R(P_n, K_{1,m}) = \begin{cases} n, & 2 \leq m \leq \lceil n/2 \rceil; \\ 2m-1, & \lceil n/2 \rceil + 1 \leq m \leq n; \\ \max\{R(P_{n-1}, K_{1,m}), R(P_n, K_{1,m-n+1}) + n - 1\}, & n \geq 3 \\ & \textit{and } m \geq n+1. \end{cases}$$

It is trivial that $R(P_2, K_{1,m}) = m + 1$. So the above recursive formula can be used to determine all path-star Ramsey numbers.

In 1978, Rousseau and Sheehan [8] gave an explicit formula for the Ramsey numbers of paths versus stars. Let $t(n, m)$, $n, m \geq 2$, be the values defined as

$$t(n, m) = \begin{cases} (n-1) \cdot \beta + 1, & \alpha \leq \gamma; \\ \lfloor (m-1)/\beta \rfloor + m, & \alpha > \gamma, \end{cases}$$

where

$$\alpha = \frac{m-1}{n-1}, \ \beta = \lceil \alpha \rceil \text{ and } \gamma = \frac{\beta^2}{\beta + 1}.$$

Theorem 2 (Rousseau and Sheehan [8]). *$R(P_n, K_{1,m}) = t(n, m)$ for all $n, m \geq 2$.*

The interested reader can compare the above two formulae. We will give an independent and short proof of Theorem 2 in Section 3.

A *linear forest* is a forest each component of which is a path. We call the graph obtained by joining a vertex to every vertex of a nontrivial linear forest a *quasar*. Thus a star is a quasar, and we call a quasar a *proper* one if it is not a star.

It may be interesting to consider the Ramsey numbers of paths versus proper quasars. Some results of this area were obtained. Salman and Broersma [9, 10] studied the Ramsey numbers of P_n versus $K_1 \vee mK_2$ (this graph is called a *fan* in [9]), and of P_n versus $K_1 \vee P_m$ (this graph is called a *kipas* in [10]). Both cases have not been completely solved in [9, 10]. Note that fans and kipases are special cases of quasars. In the following, we will consider the Ramsey numbers of paths versus proper quasars. As an application of our results, we will give a complete solution to the problem of determining the Ramsey numbers of paths versus fans.

We first determine the exact values of $R(P_n, K_1 \vee F)$ when $m \leq n$ or $m \geq 2n$, where F is a non-empty linear forest on m vertices.

Theorem 3. *Let F be a non-empty linear forest on m vertices. Then*

$$R(P_n, K_1 \vee F) = \begin{cases} 2n-1, & 2 \le m \le n; \\ t(n,m), & n \ge 2 \text{ and } m \ge 2n. \end{cases}$$

So we have an open problem for the case $n+1 \le m \le 2n-1$. For this case we have the following upper and lower bounds. By $\mathrm{par}(m)$ we denote the parity of m.

Theorem 4. *If $n \ge 2$ and $n+1 \le m \le 2n-1$, and F is a non-empty linear forest on m vertices, then*

(1) $R(P_n, K_1 \vee F) \le m+n-2+\mathrm{par}(m)$; and

(2) $R(P_n, K_1 \vee F) \ge \max\{2n-1, \lceil 3m/2 \rceil - 1, m+n-o(F)-2\}$.

If F contains no odd component, then the upper bound and the lower bound in Theorem 4 are equal. Thus we conclude the following.

Corollary 1. *If $n \ge 2$ and $n+1 \le m \le 2n-1$, and F is a linear forest on m vertices such that each component of F has an even order, then*

$$R(P_n, K_1 \vee F) = m+n-2.$$

Note that Theorem 3 and Corollary 1 give all the path-quasar Ramsey numbers $R(P_n, K_1 \vee F)$ when $o(F) = 0$, including all the Ramsey numbers of paths versus fans.

Corollary 2.

$$R(P_n, K_1 \vee mK_2) = \begin{cases} 2n-1, & 1 \le m \le \lfloor n/2 \rfloor; \\ 2m+n-2, & \lfloor n/2 \rfloor + 1 \le m \le n-1; \\ t(n,2m), & n \ge 2 \text{ and } m \ge n. \end{cases}$$

We propose the following conjecture to complete this section.

Conjecture 1. If $n \ge 2$ and $n+1 \le m \le 2n-1$, and F is a non-empty linear forest on m vertices, then

$$R(P_n, K_1 \vee F) = \max\left\{2n-1, \left\lceil \frac{3m}{2} \right\rceil - 1, m+n-o(F)-2\right\}.$$

2. Preliminaries. The following useful result is deduced from Dirac [3]. We present it here without a proof.

Theorem 5. *Every connected graph G contains a path of order at least $\min\{\nu(G), 2\delta(G)+1\}$.*

We follow the notation in [6]. For integers s, t, the *interval* $[s,t]$ is the set of integers i with $s \le i \le t$. Note that if $s > t$, then $[s,t] = \emptyset$. Let X be a subset of $\mathbb{N}$. We set $\mathcal{L}(X) = \{\sum_{i=1}^{k} x_i : x_i \in X, k \in \mathbb{N}\}$, and suppose $0 \in \mathcal{L}(X)$ for any set X. Note that if $1 \in X$, then $\mathcal{L}(X) = \mathbb{N}$. For an interval $[s,t]$, we use $\mathcal{L}[s,t]$ instead of $\mathcal{L}([s,t])$.

The following lemma was proved by the authors in [6]. We include the proof here for the completeness of our discussion.

Lemma 1. $t(n,m) = \min\{t : t \notin \mathcal{L}[t-m+1, n-1]\}$.

Proof. Set $T = \{t : t \in \mathcal{L}[t-m+1, n-1]\}$. Note that if $t \in T$, then $t-1 \in T$. So it is sufficient to prove that $t(n,m) = \max(T)+1$.

Note that

$$\begin{aligned} t \in T &\Leftrightarrow t \in \mathcal{L}[t-m+1, n-1] \\ &\Leftrightarrow t \in [k(t-m+1), k(n-1)], \text{ for some integer } k \\ &\Leftrightarrow t \le \frac{k}{k-1}(m-1) \text{ and } t \le k(n-1), \text{ for some integer } k \\ &\Leftrightarrow t \le k(n-1) \text{ for some integer } k < \alpha+1, \text{ or} \\ &\quad\ t \le \left\lfloor \frac{m-1}{k-1} \right\rfloor + m - 1, \text{ for some integer } k \ge \alpha+1. \end{aligned}$$

This implies that

$$T = \{t : t \le k(n-1), k \le \beta\} \cup \left\{ t : t \le \left\lfloor \frac{m-1}{k-1} \right\rfloor + m - 1, k \ge \beta+1 \right\}.$$

Thus

$$\begin{aligned} \max(T) &= \max\left\{ (n-1)\beta, \left\lfloor \frac{m-1}{\beta} \right\rfloor + m - 1 \right\} \\ &= \begin{cases} (n-1)\cdot\beta, & \alpha \le \gamma; \\ \lfloor (m-1)/\beta \rfloor + m - 1, & \alpha > \gamma. \end{cases} \end{aligned}$$

We conclude that $t(n,m) = \max(T)+1$. □

We use C_m to denote the cycle on m vertices, and W_m to denote the wheel on $m+1$ vertices, i.e., the graph obtained by joining a vertex to every vertex of a C_m. We will use the following formulas for path-cycle Ramsey numbers and for path-wheel Ramsey numbers.

Theorem 6 (Faudree et al. [4]). *If $n \ge 2$ and $m \ge 3$, then*

$$R(P_n, C_m) = \begin{cases} 2n-1, & \text{for } n \ge m \text{ and } m \text{ is odd}; \\ n + m/2 - 1, & \text{for } n \ge m \text{ and } m \text{ is even}; \\ \max\{m + \lfloor n/2 \rfloor - 1, 2n-1\}, & \text{for } m > n \text{ and } m \text{ is odd}; \\ m + \lfloor n/2 \rfloor - 1, & \text{for } m > n \text{ and } m \text{ is even}. \end{cases}$$

Theorem 7.

(1) (Chen et al. [2]) If $3 \le m \le n+1$, then

$$R(P_n, W_m) = \begin{cases} 3n-2, & m \text{ is odd}; \\ 2n-1, & m \text{ is even}. \end{cases}$$

(2) (Zhang [11]) If $n+2 \le m \le 2n$, then

$$R(P_n, W_m) = \begin{cases} 3n-2, & m \text{ is odd}; \\ m+n-2, & m \text{ is even}. \end{cases}$$

(3) (Li and Ning [6]) If $n \geq 2$ and $m \geq 2n+1$, then

$$R(P_n, W_m) = t(n,m).$$

3. Proofs of the theorems. Proof of Theorem 2. Let $r = t(n,m)$. By Lemma 1, $t(n,m) = \min\{t : t \notin \mathcal{L}[t-m+1, n-1]\}$. Thus $r-1 \in \mathcal{L}[r-m, n-1]$. Let $r-1 = \sum_{i=1}^{k} r_i$, where $r_i \in [r-m, n-1]$, $1 \leq i \leq k$. Let G be a graph with k components $H_1, \ldots, H_k$ such that H_i is a clique on r_i vertices. Note that G contains no P_n since every component of G has less than n vertices; and $\overline{G}$ contains no $K_{1,m}$ since every vertex of G has less than m nonadjacent vertices. This implies that $R(P_n, K_{1,m}) \geq \nu(G)+1 = r$.

Now we will prove that $R(P_n, K_{1,m}) \leq r$. Let us assume that this inequality does not hold. Let G be a graph on r vertices such that G contains no P_n and $\overline{G}$ contains no $K_{1,m}$.

Claim 1. $m + \lfloor n/2 \rfloor \leq r \leq m+n-1$, i.e., $1 \leq m+n-r \leq \lceil n/2 \rceil$.

Proof. Let $r' = m+n-1$. Since $r'-m+1 = n$, $[r'-m+1, n-1] = \emptyset$, and $r' \notin \mathcal{L}(\emptyset) = \{0\}$, we have $r \leq r' = m+n-1$ and hence $m+n-r \geq 1$.

Now we prove that $m+n-r \leq (n+1)/2$. By Lemma 1, $r \notin \mathcal{L}[r-m+1, n-1]$. Thus $r \notin [k(r-m+1), k(n-1)]$, for every k. That is, $r \in [k(n-1)+1, (k+1)(r-m+1)-1]$, for some k. This implies that

$$r \geq k(n-1)+1 \text{ and } r \geq \frac{k+1}{k}m - 1,$$

for some $k \geq 1$.

If $m \leq (k^2n - k^2 + 2k)/(k+1)$, then

$$\begin{aligned} m+n-r &\leq \frac{k^2n - k^2 + 2k}{k+1} + n - (k(n-1)+1) \\ &= \frac{n+2k-1}{k+1} \leq \frac{n+1}{2}. \end{aligned}$$

If $m > (k^2n - k^2 + 2k)/(k+1)$, then

$$\begin{aligned} m+n-r &\leq m+n-\left(\frac{k+1}{k}m - 1\right) \\ &= n - \frac{m}{k} + 1 < n - \frac{k^2n-k^2+2k}{k(k+1)} + 1 \\ &= \frac{n+2k-1}{k+1} \leq \frac{n+1}{2}. \end{aligned}$$

Thus we have $m+n-r \leq \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$. □

Case 1. Every component of G has order less than n.

Let H_i, $1 \leq i \leq k = \omega(G)$, be the components of G. Since $r \notin \mathcal{L}[r-m+1, n-1]$, there is a component, say H_1, with order at most $r-m$. Thus

$\sum_{i=2}^{k} \nu(H_i) \geq m$. Let v be a vertex in H_1. Since v is nonadjacent to every vertex in $G - H_1$, $\overline{G}$ contains a $K_{1,m}$ with the center v, a contradiction.

Case 2. There is a component of G with order at least n.

Let H be a component of G with $\nu(H) \geq n$. If every vertex of H has degree at least $\lfloor n/2 \rfloor$, then by Theorem 5, H contains a P_n, a contradiction. Thus there is a vertex v in H with $d(v) \leq \lfloor n/2 \rfloor - 1$. Let $G' = G - v - N(v)$. Then by Claim 1,

$$\nu(G') = \nu(G) - 1 - d(v) \geq r - \left\lfloor \frac{n}{2} \right\rfloor \geq m.$$

Since v is nonadjacent to every vertex in G', $\overline{G}$ contains a $K_{1,m}$ with the center v, a contradiction.

The proof is complete. □

Proof of Theorem 3. If $m = 2$, then $K_1 \vee F$ is a triangle (recall that F is non-empty). From Theorem 6, we get that $R(P_n, C_3) = 2n - 1$.

If $3 \leq m \leq n$, then $K_1 \vee F$ is a supergraph of C_3 and a subgraph of $W_{m+\mathrm{par}(m)}$, we have

$$R(P_n, C_3) \leq R(P_n, K_1 \vee F) \leq R(P_n, W_{m+\mathrm{par}(m)}).$$

By Theorems 6 and 7, $R(P_n, C_3) = R(P_n, W_{m+\mathrm{par}(m)}) = 2n - 1$. We conclude that $R(P_n, K_1 \vee F) = 2n - 1$.

Now we deal with the case $m \geq 2n$. Note that $K_1 \vee F$ is a supergraph of $K_{1,m}$ and a subgraph of W_m. We have

$$R(P_n, K_{1,m}) \leq R(P_n, K_1 \vee F) \leq R(P_n, W_m).$$

By Theorems 2 and 7, $R(P_n, K_{1,m}) = R(P_n, W_m) = t(n, m)$ (we remark that if $m = 2n$, then $m + n - 2 = t(n, m)$). We conclude that $R(P_n, K_1 \vee F) = t(n, m)$.

The proof is complete. □

Proof of Theorem 4. Since $K_1 \vee F$ is a subgraph of $W_{m+\mathrm{par}(m)}$, by Theorem 7, we have

$$R(P_n, K_1 \vee F) \leq m + n - 2 + \mathrm{par}(m).$$

Now we construct three graphs. Let

$$\begin{aligned} G_1 &= 2K_{n-1}, \\ G_2 &= K_{\lfloor m/2 \rfloor} \cup 2K_{\lceil m/2 \rceil - 1} \text{ and} \\ G_3 &= K_{n-1} \cup 2K_{(m-o(F))/2-1}. \end{aligned}$$

One can check that all the three graphs contain no P_n and their complements contain no $K_1 \vee F$. This implies that $R(P_n, K_1 \vee F) \geq \max\{\nu(G_i) + 1 :$

$i = 1, 2, 3\}$. Since $\nu(G_1) = 2n - 2$, $\nu(G_2) = \lceil 3m/2 \rceil - 2$ and $\nu(G_3) = m + n - o(F) - 3$, we get that

$$R(P_n, K_1 \vee F) \geq \max\left\{2n-1, \left\lceil \frac{3m}{2} \right\rceil - 1, m+n-o(F)-2\right\}.$$

The proof is complete. □

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