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Rotation indices related to Poncelet's closure theorem

Abstract. Let $C_R C_r$ denote an annulus formed by two non-concentric circles C_R, C_r in the Euclidean plane. We prove that if Poncelet's closure theorem holds for k-gons circuminscribed to $C_R C_r$, then there exist circles inside this annulus which satisfy Poncelet's closure theorem together with C_r, with n-gons for any $n > k$.

1. Introduction. Poncelet's closure theorem, going back to the 19th century, has various interesting forms and applications; cf. [2], [7], [4], [9], and the excellent survey [3] as well as [4]. The rich history of this theorem is presented in [1, Ch. 16], [8, § 2.4], and [7], and our paper refers to circular versions of it. Let C_R, C_r be two circles with radii $R > r > 0$ and C_r lying inside C_R. From any point on C_R, draw a tangent to C_r and extend it to C_R again, using the obtained new intersection point with C_R for starting with a new tangent to C_r, etc.; the system of tangential segments obtained in this way inside C_R is called a Poncelet transverse (or bar billiard). We say that the annulus $C_R C_r$ has *Poncelet's porism property* if there is a starting point on C_R for which a Poncelet traverse is a closed polygon. *Poncelet's closure theorem* (for circles) says that then the transverse will also close for any other starting point from C_R. It is known that such closing polygons (with or without self-intersections) correspond to rational rotations; e.g.,

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the rotation number or *index* $\frac{1}{3}$ is related to a triangle "between" C_R and C_r, and the index $\frac{2}{5}$ to a (self-intersecting) pentagram.

In [6] it was proved that "close" to a pair of circles, which have Poncelet's porism property for index $\frac{1}{3}$, there exist unique pairs of circles having this property with respect to indices $\frac{1}{4}$ and $\frac{1}{6}$, and it was conjectured there that this holds true for arbitrary indices.

In the present paper we show that this conjecture is true in the following sense: for a pair of circles having Poncelet's porism property for index $\frac{1}{k}$, with $k \geq 3$ as natural number, we prove that there exists a circle lying between the starting circles such that this circle together with the smaller given circle has Poncelet's porism property for any given index $\frac{1}{n}$, where n is an arbitrary natural number with $n > k$.

2. Basic notions and tools. Let us consider a circular annulus $C_rC_{a,R}$ formed by two circles C_r and $C_{a,R}$. The circles C_r and $C_{a,R}$ are given by the equations $x^2 + y^2 = r^2$ and $(x-a)^2 + y^2 = R^2$, respectively, with

$$0 < a < R - r. \tag{1}$$

Recall the following form of Poncelet's closure theorem which is suitable for our purpose; see [1].

If there exists a one circuminscribed (*i.e., simultaneously inscribed in the outer circle and circumscribed about the inner circle*) *n-gon in a circular annulus, then any point of the outer circle is the vertex of some circuminscribed n-gon.*

If Poncelet's closure theorem holds for $n = 3$, then Euler's condition

$$R^2 - 2Rr - a^2 = 0 \tag{2}$$

is satisfied. We will denote this condition by $\mathrm{Pct}\,(C_rC_{a,R}, 3)$. There is no elementary formula for the analogously defined condition $\mathrm{Pct}\,(C_rC_{a,R}, n)$, but we note that $\mathrm{Pct}\,(C_rC_{a,R}, 4)$ and $\mathrm{Pct}\,(C_rC_{a,R}, 6)$ have the forms

$$\left(R^2 - a^2\right)^2 = 2r^2\left(R^2 + a^2\right) \tag{3}$$

and

$$3\left(R^2 - a^2\right)^4 = 4r^2\left(R^2 + a^2\right)\left(R^2 - a^2\right)^2 + 16r^2a^2R^2, \tag{4}$$

respectively; see [3].

It is amazing that for particular natural numbers we have elementary conditions involving also radicals, while for an arbitrary natural number $n \geq 3$ only the Jacobi formula (cf. formula (7) in [10]), using elliptic functions, is involved.

For further use we introduce a convenient parametrization of the annulus $C_rC_{a,R}$. Namely, we take the parametrization $z\,(t) = re^{it}$ for C_r, and for $C_{a,R}$ we use

$$w\,(t) = z\,(t) + \lambda\,(t)\,ie^{it}, \qquad t \in [0, 2\pi]\,, \tag{5}$$

where $\lambda(t) = \sqrt{R^2 - (r - a\cos t)^2} - a\sin t$.

The line which is tangent to the circle C_r at a point $z(t)$ intersects the circle C_R at a point $w(t) = z(t) + \lambda(t) i e^{it}$. Let us draw a second tangent line to C_r, passing at $w(t)$. It intersects C_r at a point $z(\varphi(t))$, where $\varphi(t)$ satisfies the condition

$$\tan\frac{\varphi(t) - t}{2} = \frac{\lambda(t)}{r}. \tag{6}$$

In [5] it is proved that

$$\varphi' = \frac{\sqrt{1 - (\sigma\circ\varphi)^2}}{\sqrt{1-\sigma^2}}, \tag{7}$$

where

$$\sigma(t) = \frac{r - a\cos t}{R}. \tag{8}$$

It is routine to check that the solution of this differential equation with initial condition $\varphi(0) = m$ is given by the formula

$$\varphi(t) = B^{-1}\left(B(t) + B(m)\right), \tag{9}$$

where

$$B(t) = \int_0^t \frac{ds}{\sqrt{1 - \sigma^2(s)}}. \tag{10}$$

3. Results and proofs.

Theorem 1. *Poncelet's closure theorem holds in the annulus $C_r C_{a,R}$ for n-gons, $n \geq 3$, if and only if the following identity holds:*

$$B\left(t + 2\arctan\frac{\lambda(t)}{r}\right) \equiv B(t) + \frac{1}{n}B(2\pi). \tag{11}$$

Proof. $\Rightarrow$) From the assumption it follows that Poncelet's transverse closes after n reflections, forming a circuminscribed convex n-gon. This is equivalent to the condition

$$\varphi^{[n]}(t) = t + 2\pi \qquad \text{for all } t \in \mathbb{R}, \tag{12}$$

where

$$\varphi^{[1]} = \varphi \quad \text{and} \quad \varphi^{[n+1]} = \varphi^{[n]}\circ\varphi \qquad \text{for } n = 1, 2, 3, \ldots \tag{13}$$

Note that formula (9) implies

$$\varphi^{[n]}(t) = B^{-1}\left(B(t) + nB(m)\right). \tag{14}$$

From (12) and (14) it follows immediately that

$$B(2\pi) = nB(m). \tag{15}$$

Finally, the function φ is given by the formula

$$\varphi(t) = B^{-1}\left(B(t) + \frac{1}{n}B(2\pi)\right), \tag{16}$$

and

$$\varphi(0) = m = B^{-1}\left(\frac{1}{n}B(2\pi)\right). \tag{17}$$

From (6) we get

$$\varphi(t) = t + 2\arctan\frac{\lambda(t)}{r}. \tag{18}$$

The formulas (17) and (18) imply the identity (11).
$\Leftarrow$) Assume that in the annulus $C_r C_{a,R}$ the identity (11) holds for some natural number $n \geq 3$. From the formulas (10) and (16) we get

$$\varphi^{[n]}(t) = B^{-1}(B(t) + B(2\pi)) = B^{-1}(B(t + 2\pi)) = t + 2\pi.$$

□

Now, using (10), we can rewrite the identity (11) in the form

$$\int_0^{t+2\arctan\frac{\lambda(t)}{r}} \frac{1}{\sqrt{1-\sigma^2(s)}}ds \equiv \int_0^t \frac{1}{\sqrt{1-\sigma^2(s)}}ds + \frac{1}{n}\int_0^{2\pi} \frac{1}{\sqrt{1-\sigma^2(s)}}ds. \tag{19}$$

Hence we have

$$\int_t^{2\arctan\frac{\lambda(t)}{r}} \frac{1}{\sqrt{1-\sigma^2(s)}}ds \equiv \frac{1}{n}\int_0^{2\pi} \frac{1}{\sqrt{1-\sigma^2(s)}}ds. \tag{20}$$

In the particular case $t = 0$ we have

$$\int_0^{2\arctan\frac{1}{r}\sqrt{R^2-(r-a)^2}} \frac{1}{\sqrt{1-\sigma^2(s)}}ds = \frac{1}{n}\int_0^{2\pi} \frac{1}{\sqrt{1-\sigma^2(s)}}ds. \tag{21}$$

This is exactly the formula (5.6) from [5], and we note that it implies Poncelet's porism property for n-gons.

Introducing

$$V_\xi = \frac{1}{r}\sqrt{\left[(1-\xi)r + \xi R\right]^2 - (r - \xi a)^2} \tag{22}$$

for $\xi \in [0,1]$, we have

$$V_\xi = \frac{1}{r}\sqrt{(R - r + a)\left[(R - r - a)\xi^2 + 2r\xi\right]}. \tag{23}$$

Since $0 < a < R - r$, we can write

$$V_\xi = \frac{1}{r} c(\xi) \sqrt{R - r + a} \qquad \text{for } \xi \in [0, 1], \tag{24}$$

where

$$c(\xi) = \sqrt{(R - r - a)\xi^2 + 2r\xi}. \tag{25}$$

Note that

$$V_1 = \frac{1}{r} \sqrt{R^2 - (r - a)^2} \qquad \text{and} \qquad V_0 = 0. \tag{26}$$

Similarly, we define

$$\sigma_\xi(t) = \frac{r - \xi a \cos t}{(1 - \xi) r + \xi R} \qquad \text{for } \xi \in [0, 1], \tag{27}$$

and one has $\sigma_1 = \sigma$ and $\sigma_0 = 1$.

Now we will prove our main theorem.

Theorem 2. *Assume that Poncelet's closure theorem holds in an annulus $C_r C_{a,R}$ for k-gons, $k \geq 3$. Then for any $n > k$ there exists $\gamma \in (0, 1)$ such that Poncelet's closure theorem holds in the annulus $C_r C_{\gamma a, (1-\gamma) r + \gamma R}$ for n-gons.*

Proof. Using the equality (20) from the proof of Theorem 1, we introduce the function

$$F_n(\xi) = n \int_0^{2 \arctan V_\xi} \frac{1}{\sqrt{1 - \sigma_\xi^2(s)}} ds - \int_0^{2\pi} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds. \tag{28}$$

First we have

$$F_n(1) = n \int_0^{2 \arctan V_1} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds - \int_0^{2\pi} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds.$$

From now on we assume that the starting annulus $C_r C_{a,R}$ has Poncelet's porism property for a natural number $k \geq 3$, and we consider $n > k$. Then by (20) we have

$$k \int_0^{2 \arctan V_1} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds = \int_0^{2\pi} \frac{1}{\sqrt{1 - \sigma^2(s)}} ds. \tag{29}$$

Using this condition, we get

$$F_n(1) = (n-k)\int_0^{2\arctan V_1} \frac{1}{\sqrt{1-\sigma^2(s)}}ds + k\int_0^{2\arctan V_1} \frac{1}{\sqrt{1-\sigma^2(s)}}ds$$
$$-\int_0^{2\pi} \frac{1}{\sqrt{1-\sigma^2(s)}}ds = (n-k)\int_0^{2\arctan V_1} \frac{1}{\sqrt{1-\sigma^2(s)}}ds > 0.$$

In order to evaluate $F_n(0)$, we first calculate the value $F_n(\varepsilon)$ for $\varepsilon \in (0,1)$. We have

$$F_n(\varepsilon) = n\int_0^{2\arctan V_\epsilon} \frac{1}{\sqrt{1-\sigma_\varepsilon^2(s)}}ds - \int_0^{2\pi} \frac{1}{\sqrt{1-\sigma_\varepsilon^2(s)}}ds$$
$$= (n-1)\int_0^{2\arctan V_\epsilon} \frac{1}{\sqrt{1-\sigma_\varepsilon^2(s)}}ds - \int_{2\arctan V_\epsilon}^{2\pi} \frac{1}{\sqrt{1-\sigma_\varepsilon^2(s)}}ds.$$

First we prove that

$$\lim_{\varepsilon\to 0^+}\int_0^{2\arctan V_\epsilon} \frac{1}{\sqrt{1-\sigma_\varepsilon^2(s)}}ds \le C, \tag{30}$$

for some positive constant C. We calculate

$$\int_0^{2\arctan V_\epsilon} \frac{1}{\sqrt{1-\sigma_\varepsilon^2(s)}}ds$$
$$= \int_0^{2\arctan \frac{1}{r}c(\varepsilon)\sqrt{R-r+a}} \left[1-\left(\frac{r-a\varepsilon\cos t}{(1-\varepsilon)r+\varepsilon R}\right)^2\right]^{-\frac{1}{2}} dt$$
$$= \int_0^{2\arctan \frac{1}{r}c(\varepsilon)\sqrt{R-r+a}} \left(\frac{[(1-\varepsilon)r+\varepsilon R]^2-(r-\varepsilon a\cos t)^2}{((1-\varepsilon)r+\varepsilon R)^2}\right)^{-\frac{1}{2}} dt$$
$$= \int_0^{2\arctan \frac{1}{r}c(\varepsilon)\sqrt{R-r+a}} \frac{(1-\varepsilon)r+\varepsilon R}{\sqrt{(R-r+a\cos t)\left[(R-r-a\cos t)\varepsilon^2+2r\varepsilon\right]}}dt$$

$$\leq \int_0^{2\arctan\frac{1}{r}c(\varepsilon)\sqrt{R-r+a}} \frac{(1-\varepsilon)\,r+\varepsilon R}{\sqrt{(R-r-a)\left[(R-r-a)\,\varepsilon^2+2r\varepsilon\right]}}dt$$

$$= \left[(1-\varepsilon)\,r+\varepsilon R\right] \int_0^{2\arctan\frac{1}{r}c(\varepsilon)\sqrt{R-r+a}} \frac{1}{c\,(\varepsilon)\sqrt{R-r-a}}dt$$

$$= \left[(1-\varepsilon)\,r+\varepsilon R\right] \frac{2\arctan\frac{1}{r}c\,(\varepsilon)\sqrt{R-r+a}}{c\,(\varepsilon)\sqrt{R-r+-a}}.$$

Since $\arctan x < x$ for $x > 0$, then

$$\int_0^{2\arctan V_\epsilon} \frac{1}{\sqrt{1-\sigma_\varepsilon^2\,(s)}}ds \leq \frac{2}{r}\left[(1-\varepsilon)\,r+\varepsilon R\right]\frac{\sqrt{R-r+a}}{\sqrt{R-r-a}}. \tag{31}$$

Thus

$$\lim_{\varepsilon\to 0^+}\int_0^{2\arctan V_\epsilon} \frac{1}{\sqrt{1-\sigma_\varepsilon^2\,(s)}}ds \leq C = \frac{2}{r}\frac{\sqrt{R-r+a}}{\sqrt{R-r-a}}. \tag{32}$$

Next, we claim that

$$\lim_{\varepsilon\to 0^+}\int_{2\arctan V_\epsilon}^{2\pi} \frac{1}{\sqrt{1-\sigma_\varepsilon^2\,(s)}}ds = +\infty. \tag{33}$$

We have

$$\begin{aligned} &\int_{2\arctan V_\epsilon}^{2\pi} \frac{1}{\sqrt{1-\sigma_\varepsilon^2\,(s)}}ds \\ &= \int_{2\arctan V_\epsilon}^{2\pi} \frac{(1-\varepsilon)\,r+\varepsilon R}{\sqrt{R-r+a\cos t}\cdot\sqrt{(R-r-a\cos t)\,\varepsilon^2+2r\varepsilon}}dt \end{aligned} \tag{34}$$

and, furthermore,

$$\begin{aligned} &((1-\varepsilon)\,r+\varepsilon R)\int_{2\arctan V_\epsilon}^{2\pi} \frac{1}{\sqrt{R-r+a}\cdot\sqrt{(R-r+a)\,\varepsilon^2+2r\varepsilon}}dt \\ &= \frac{(1-\varepsilon)\,r+\varepsilon R}{\sqrt{R-r+a}}\cdot\frac{2\pi-2\arctan\frac{1}{r}\sqrt{R-r+a}\cdot c\,(\varepsilon)}{\sqrt{(R-r+a)\,\varepsilon^2+2r\varepsilon}} \longrightarrow +\infty\,, \end{aligned}$$

when $\varepsilon \to 0$. Hence

$$\lim_{\varepsilon\to 0^+}\int_{2\arctan V_\epsilon}^{2\pi} \frac{1}{\sqrt{1-\sigma_\varepsilon^2\,(s)}}ds = +\infty\,. \tag{35}$$

Thus, we have

$$F_n\left(0^+\right) = \lim_{\varepsilon\to 0^+} F_n(\varepsilon) = -\infty \tag{36}$$

and

$$F_n(1) > 0.$$

These conditions imply that there exists a number $\gamma \in (0,1)$ such that

$$F_n(\gamma) = 0. \tag{37}$$

Thus, with Theorem 1 the proof is finished. □

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