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The natural transformations between r-tangent and r-cotangent bundles over Riemannian manifolds

Dedicated to Professor Andrzej Zajtz on the occasion of his 80th birthday with respect and gratitude.

Abstract. If (M, g) is a Riemannian manifold, we have the well-known base preserving vector bundle isomorphism $TM \tilde{=} T^*M$ given by $v \to g(v, -)$ between the tangent TM and the cotangent T^*M bundles of M. In the present note, we generalize this isomorphism to the one $T^{(r)}M \tilde{=} T^{r*}M$ between the r-th order vector tangent $T^{(r)}M = (J^r(M, \mathbf{R})_0)^*$ and the r-th order cotangent $T^{r*}M = J^r(M, \mathbf{R})_0$ bundles of M. Next, we describe all base preserving vector bundle maps $C_M(g) : T^{(r)}M \to T^{r*}M$ depending on a Riemannian metric g in terms of natural (in g) tensor fields on M.

0. All manifolds are assumed to be smooth, Hausdorff, finite dimensional and without boundaries. Maps are assumed to be smooth (of class C^∞). The category of m-dimensional manifolds and their embeddings is denoted by $\mathcal{M}f_m$.

It is clear that the tangent TM and the cotangent T^*M bundles of M are not canonically isomorphic. However, if g is a Riemannian metric on a manifold M, we have the base preserving vector bundle isomorphism i_g : $TM \to T^*M$ given by $i_g(v) = g(v, -)$, $v \in T_xM$, $x \in M$.

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In the present note we generalize the isomorphism $TM \tilde{=} T^*M$ depending on g, to a base preserving vector bundle isomorphism $i_g^{<r>} : T^{(r)}M \to T^{r*}M$ canonically depending on g between the r-th order vector tangent bundle $T^{(r)}M = (J^r(M, \mathbf{R})_0)^*$ and the r-th order cotangent bundle $T^{r*}M = J^r(M, \mathbf{R})_0$ of M. Next, we study the problem of describing all $\mathcal{M}f_m$-natural operators $C : Riem \rightsquigarrow Hom(T^{(r)}, T^{r*})$ transforming Riemannian metrics g on m-dimensional manifolds M into base preserving vector bundle maps $C_M(g) : T^{(r)}M \to T^{r*}M$. We prove that this problem can be reduced to the (partially well-known) one of describing all $\mathcal{M}f_m$-natural operators $t : Riem \rightsquigarrow S^lT^* \otimes S^kT^*$ (for $l, k = 1, \dots, r$) sending Riemannian metric on M into tensor fields $t_M(g)$ of types $S^lT^* \otimes S^kT^*$ on M.

The r-th order cotangent bundle is a functor $T^{r*} : \mathcal{M}f_m \to \mathcal{VB}$ sending any m-manifold M into $T^{r*}M := J^r(M, \mathbf{R})_0$ (the vector bundle of r-jets $M \to \mathbf{R}$ with target 0) and any embedding $\varphi : M_1 \to M_2$ of two m-manifolds into $T^{r*}\varphi : T^{r*}M_1 \to T^{r*}M_2$ given by $T^{r*}\varphi(j^r_x\gamma) = j^r_{\varphi(x)}(\gamma \circ \varphi^{-1})$, $j^r_x\gamma \in T^{r*}M$. If $r = 1$, $T^{1*}M \tilde{=} T^*M$ (the usual cotangent bundle) by $j^1_x\gamma \tilde{=} d_x\gamma$.

The r-th order vector tangent bundle $T^{(r)} : \mathcal{M}f_m \to \mathcal{VB}$ is the natural bundle dual to the r-th order cotangent bundle, i.e. $T^{(r)}M = (T^{r*}M)^*$ and $T^{(r)}\varphi = (T^{r*}\varphi^{-1})^*$.

The concept of natural bundles can be found in [3], [7], [8]. Natural constructions on $T^{(r)}M$ has been studied e.g. in [6].

A general definition of natural operators can be found in [3]. In our note, an $\mathcal{M}f_m$-natural operator $C : Riem \rightsquigarrow Hom(T^{(r)}, T^{r*})$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $C_M(g) : T^{(r)}M \to T^{r*}M$ is an $\mathcal{M}f_m$-invariant system $C = \{C_M\}_{M \in obj(\mathcal{M}f_m)}$ of regular operators (functions)

$$C_M : Riem(M) \to Hom_M(T^{(r)}M, T^{r*}M)$$

for any m-manifold M, where $Riem(M)$ is the set of all Riemannian metrics on M and $Hom_M(T^{(r)}M, T^{r*}M)$ is the set of all vector bundle maps $T^{(r)}M \to T^{r*}M$ covering id_M. More precisely, the $\mathcal{M}f_m$-invariance of C means that if $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ are φ-related by an embedding $\varphi : M_1 \to M_2$ of m-manifolds (i.e. φ is (g_1, g_2)-isomorphism), then $C_{M_1}(g_1)$ and $C_{M_2}(g_2)$ are φ-related (i.e. $C_{M_2}(g_2) \circ T^{(r)}\varphi = T^{r*}\varphi \circ C_{M_1}(g_1)$). The regularity means that C_M transforms smoothly parametrized families of Riemannian metrics into smoothly parametrized ones of vector bundle maps.

Similarly, an $\mathcal{M}f_m$-natural operator (natural tensor) $t : Riem \rightsquigarrow \otimes^pT \otimes \otimes^qT^*$ is an $\mathcal{M}f_m$-invariant system $t = \{t_M\}_{M \in obj(\mathcal{M}f_m)}$ of regular operators

$$t_M : Riem(M) \to \mathcal{T}^{p,q}(M)$$

for any $M \in \mathcal{M}f_m$, where $\mathcal{T}^{p,q}(M)$ is the set of tensor fields of type (p,q) on M.

An explicit example of a natural operator $C : Riem \rightsquigarrow Hom(T^{(r)}, T^{r*})$ will be presented in item 1.

A full description of all polynomial natural tensors $t : Riem \rightsquigarrow \otimes^p T^* \otimes \otimes^q T$ can be found in [1]. This description is as follows. Each covariant derivative of the curvature $\mathcal{R}(g) \in \mathcal{T}^{(0,4)}(M)$ of a Riemannian metric g is a natural tensor and g is a natural tensor. Further every tensor multiplication of two natural tensors and every contraction on one covariant and one contravariant entry of a natural tensor give new natural tensor. Finally, we can tensor any natural tensor with a metric independent natural tensor, we can permute any number of entries in the tensor product and we can repeat these steps and take linear combinations. In this way we can obtain any natural tensor of types (p,q) depending polynomially on a Riemannian metric.

Taking respective type natural tensors and applying respective symmetrization, we can produce many natural tensors $t : Riem \rightsquigarrow S^l T^* \otimes S^k T^*$.

1. We are going to present an example of an $\mathcal{M}f_m$-natural operator $C : Riem \rightsquigarrow Hom(T^{(r)}, T^{r*})$. We start with some preparations.

It is well known (see [2]) that if g is a Riemannian tensor field on a manifold M and $x \in M$, then there is a g-normal coordinate system $\varphi : (M,x) \to (\mathbf{R}^m, 0)$ with center x. If $\psi : (M,x) \to (\mathbf{R}^m, 0)$ is another g-normal coordinate system with center x, then there is $A \in O(m)$ such that $\psi = A \circ \varphi$ near x.

We have the following important proposition.

Proposition 1. *Let g be a Riemannian tensor field on a manifold M. Then there are (canonical in g) vector bundle isomorphisms*

$$I_g : T^{r*}M \to \oplus_{k=1}^r S^k T^* M \text{ and } J_g : T^{(r)}M \to \oplus_{k=1}^r S^k TM$$

covering the identity map of M.

Proof. Let $v \in T^{r*}_x M$, $x \in M$. Let $\varphi : (M,x) \to (\mathbf{R}^m, 0)$ be a g-normal coordinate system with center x. We put

$$I_g(v) = I_g^{\varphi}(v) := \oplus_{k=1}^r S^k T^* \varphi^{-1} \circ I \circ T^{r*}\varphi(v) \ ,$$

where $I : T_0^{r*}\mathbf{R}^m = J_0^r(\mathbf{R}^m, \mathbf{R})_0 \to \oplus_{k=1}^r S^k T_0^* \mathbf{R}^m = \oplus_{k=1}^r S^k \mathbf{R}^{m*}$ is the obvious $O(m)$-invariant vector space isomorphism. If $\psi : (M,x) \to (\mathbf{R}^m, 0)$ is another g-normal coordinate system with center x, then $\psi = A \circ \varphi$ (near x) for some $A \in O(m)$. Using the $O(m)$-invariance of I, we deduce that $I_g^{\psi}(v) = I_g^{\varphi}(v)$. So, the definition of $I_g(v)$ is independent of the choice of φ. So, isomorphism $I_g : T^{r*}M \to \oplus_{k=1}^r S^k T^* M$ is well-defined.

Quite similarly, one can define isomorphism $J_g : T^{(r)}M \to \oplus_{k=1}^r S^k TM$. □

Example 1. Given a Riemannian metric g on a manifold M, we have isomorphism

$$i_g : TM \tilde{=} T^*M \ , \ i_g(v) = g(v, -) \ ,$$

and then we have (obtained in obvious way from i_g) the base preserving vector bundle isomorphism

$$i_g^{(r)} : \oplus_{k=1}^r S^k TM \tilde{=} \oplus_{k=1}^r S^k T^*M \ , \ i_g^{(r)}(v_1 \odot \ldots \odot v_k) = i_g(v_1) \odot \ldots \odot i_g(v_k) \, .$$

Now, using the base preserving vector bundle isomorphisms J_g and I_g (from Proposition 1), we get the base preserving vector bundle isomorphism

$$i_g^{<r>} = I_g \circ i_g^{(r)} \circ J_g^{-1} : T^{(r)}M \to T^{r*}M \ .$$

Thus the family $C^{(r)} : Riem \rightsquigarrow Hom(T^{(r)}, T^{r*})$ of operators

$$C_M^{(r)} : Riem(M) \to Hom_M(T^{(r)}M), T^{r*}M) \ , \ C_M^{(r)}(g) = i_g^{<r>}$$

for all $M \in obj(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$-natural operator.

2. Let $g \in Riem(M)$ be a Riemannian metric on an m-manifold M. By Proposition 1 and Example 1,

$$T^{(r)}M = T^{r*}M = \oplus_{k=1}^r S^k T^*M = \oplus_{k=1}^r S^k TM$$

modulo the base preserving vector bundle isomorphisms canonically depending on g. Consequently, our problem of finding all $\mathcal{M}f_m$-natural operators $C : Riem \rightsquigarrow Hom(T^{(r)}, T^{r*})$ is reduced to the one of finding all systems $(C^{l,k})$ of $\mathcal{M}f_m$-natural operators

$$C^{l,k} : Riem \rightsquigarrow Hom(S^l T, S^k T^*)$$

transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $C_M^{l,k}(g) : S^l TM \to S^k T^*M$, where $l, k = 1, \ldots, r$, or (equivalently) our problem is reduced to the one of finding all natural tensors $C^{l,k} : Riem \rightsquigarrow S^l T^* \otimes S^k T^*$, $l, k = 1, \ldots, r$.

Thus we have proved the following theorem.

Theorem 1. *The $\mathcal{M}f_m$-natural operators $C : Riem \rightsquigarrow Hom(T^{(r)}, T^{r*})$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $C_M(g) : T^{(r)}M \to T^{r*}M$ are in bijection with the systems $(C^{l,k})$ of $\mathcal{M}f_m$-natural operators (natural tensors) $C^{l,k} : Riem \rightsquigarrow S^l T^* \otimes S^k T^*$ for $l, k = 1, \ldots, r$.*

This result is interesting because any natural transformation $T^{(r)}M \to T^{r*}M$ is the zero one.

Now, using the isomorphism $T^{(r)}M \tilde{=} T^{r*}M$ depending on g, we have the following corollary of Theorem 1.

Corollary 1. *The $\mathcal{M}f_m$-natural operators $C : Riem \rightsquigarrow Hom(T^{r*}, T^{(r)})$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $C_M(g) : T^{r*}M \to T^{(r)}M$ are in bijection with the systems $(C^{l,k})$ of natural tensors $C^{l,k} : Riem \rightsquigarrow S^lT^* \otimes S^kT^*$ for $l, k = 1, \dots, r$.*

This result is interesting because any natural transformation $T^{r*}M \to T^{(r)}M$ is the zero one, too.

By the same reason, we have also the following corollary.

Corollary 2. *The $\mathcal{M}f_m$-natural operators $C : Riem \rightsquigarrow Hom(T^{(r)}, T^{(r)})$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $C_M(g) : T^{(r)}M \to T^{(r)}M$ are in bijection with the systems $(C^{l,k})$ of natural tensors $C^{l,k} : Riem \rightsquigarrow S^lT^* \otimes S^kT^*$ for $k, l = 1, \dots, r$.*

This result is interesting because of the result of I. Kolář and G. Vosmanská [4] saying that any natural transformation $T^{(r)}M \to T^{(r)}M$ is a constant multiple of the identity.

We have also the next similar corollary.

Corollary 3. *The $\mathcal{M}f_m$-natural operators $C : Riem \rightsquigarrow Hom(T^{r*}, T^{r*})$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $C_M(g) : T^{r*}M \to T^{r*}M$ are in bijection with the systems $(C^{l,k})$ of natural tensors $C^{l,k} : Riem \rightsquigarrow S^lT^* \otimes S^kT^*$ for $l, k = 1, \dots, r$.*

This result is interesting because of the result of J. Kurek [5] saying (in particular) that any vector bundle natural transformation $T^{r*}M \to T^{r*}M$ is a constant multiple of the identity.

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