

D. VAMSHEE KRISHNA<sup>1</sup>, B. VENKATESWARLU  
and T. RAMREDDY

### Third Hankel determinant for starlike and convex functions with respect to symmetric points

ABSTRACT. The objective of this paper is to obtain best possible upper bound to the  $H_3(1)$  Hankel determinant for starlike and convex functions with respect to symmetric points, using Toeplitz determinants.

**1. Introduction.** Let  $A$  denote the class of functions  $f$  of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the open unit disc  $E = \{z : |z| < 1\}$ . Let  $S$  be the subclass of  $A$  consisting of univalent functions. For any two analytic functions  $g$  and  $h$  respectively with their expansions as  $g(z) = \sum_{k=0}^{\infty} a_k z^k$  and  $h(z) = \sum_{k=0}^{\infty} b_k z^k$ , the Hadamard product or convolution of  $g(z)$  and  $h(z)$  is defined as the power series

$$(g * h)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

---

<sup>1</sup>Corresponding author

2010 *Mathematics Subject Classification.* 30C45, 30C50.

*Key words and phrases.* Analytic function, starlike and convex functions with respect to symmetric points, upper bound, Hankel determinant, convolution, positive real function, Toeplitz determinants.

The Hankel determinant of  $f$  for  $q \geq 1$  and  $n \geq 1$  was defined by Pommerenke [9] as

$$(1.2) \quad H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1).$$

One can easily observe that the Fekete–Szegő functional is  $H_2(1)$ . Fekete–Szegő then further generalized the estimate  $|a_3 - \mu a_2^2|$  with  $\mu$  real and  $f \in S$ . Ali [1] found sharp bounds on the first four coefficients and sharp estimate for the Fekete–Szegő functional  $|\gamma_3 - t\gamma_2^2|$ , where  $t$  is real, for the inverse function of  $f$  defined as  $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ , when  $f \in \widetilde{ST}(\alpha)$ , the class of strongly starlike functions of order  $\alpha$  ( $0 < \alpha \leq 1$ ). Further, sharp bounds for the functional

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2|,$$

when  $q = 2$  and  $n = 2$ , known as the second Hankel determinant, were obtained for various subclasses of univalent and multivalent analytic functions. For our discussion, in this paper, we consider the Hankel determinant in the case of  $q = 3$  and  $n = 1$ , denoted by  $H_3(1)$ , given by

$$(1.3) \quad H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

For  $f \in A$ ,  $a_1 = 1$ , so that, we have

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

and by applying triangle inequality, we obtain

$$(1.4) \quad |H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_2 a_3 - a_4| + |a_5||a_3 - a_2^2|.$$

Babalola [2] obtained sharp upper bounds to the functional  $|a_2 a_3 - a_4|$  and  $|H_3(1)|$  for the familiar subclasses namely starlike and convex functions respectively denoted by  $ST$  and  $CV$  of  $S$ . The sharp upper bounds to the second Hankel determinant  $|a_2 a_4 - a_3^2|$  for the classes  $ST$  and  $CV$  were obtained by Janteng et al. [6].

Motivated by the results obtained by Babalola [2] and recently by Raja and Malik [11] in finding the sharp upper bound to the Hankel determinant  $|H_3(1)|$  for certain subclasses of  $S$ , in this paper, we obtain an upper bound to the functional  $|a_2 a_3 - a_4|$  and hence for  $|H_3(1)|$ , for the function  $f$  given in (1.1), belonging to the classes namely starlike with respect to symmetric points and convex with respect to symmetric points denoted by  $ST_s$  and  $CV_s$  respectively, defined as follows.

**Definition 1.1.** A function  $f(z) \in A$  is said to be in the class  $ST_s$ , if it satisfies the condition

$$(1.5) \quad \operatorname{Re} \left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad \forall z \in E.$$

The class  $ST_s$  was introduced and studied by Sakaguchi [15]. Further, he has shown that the functions in  $ST_s$  are close-to-convex and hence are univalent. The concept of starlike functions with respect to symmetric points have been extended to starlike functions with respect to  $N$ -symmetric points by Ratanchand [14] and Prithvipalsingh [10]. RamReddy [12] studied the class of close-to-convex functions with respect to  $N$ -symmetric points and proved that this class is closed under convolution with convex univalent functions.

**Definition 1.2.** A function  $f(z) \in A$  is said to be in  $CV_s$ , if it satisfies the condition

$$(1.6) \quad \operatorname{Re} \left\{ \frac{2\{zf'(z)\}'}{\{f(z) - f(-z)\}'} \right\} > 0, \quad \forall z \in E.$$

The class  $CV_s$  was introduced and studied by Das and Singh [3]. From the Definitions 1.1 and 1.2, it is evident that  $f \in CV_s$  if and only if  $zf' \in ST_s$ . Some preliminary lemmas required for proving our results are as follows:

**2. Preliminary Results.** Let  $\mathcal{P}$  denote the class of functions consisting of  $p$ , such that

$$(2.1) \quad p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

which are regular in the open unit disc  $E$  and satisfy  $\operatorname{Re} p(z) > 0$ , for any  $z \in E$ . Here  $p(z)$  is called the Carathéodory function [4].

**Lemma 2.1** ([8, 16]). *If  $p \in \mathcal{P}$ , then  $|c_k| \leq 2$ , for each  $k \geq 1$  and the inequality is sharp for the function  $\frac{1+z}{1-z}$ .*

**Lemma 2.2** ([5]). *The power series for  $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$  given in (2.1) converges in the open unit disc  $E$  to a function in  $\mathcal{P}$  if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \dots$$

and  $c_{-k} = \bar{c}_k$ , are all non-negative. They are strictly positive except for  $p(z) = \sum_{k=1}^m \rho_k p_0(e^{it_k} z)$ , with  $\sum_{k=1}^m \rho_k = 1$ ,  $t_k$  real and  $t_k \neq t_j$ , for  $k \neq j$ ,

where  $p_0(z) = \frac{1+z}{1-z}$ ; in this case  $D_n > 0$  for  $n < (m-1)$  and  $D_n = 0$  for  $n \geq m$ .

This necessary and sufficient condition found in [5] is due to Carathéodory and Toeplitz. We may assume without restriction that  $c_1 > 0$ . On using Lemma 2.2, for  $n = 2$ , we have

$$(2.2) \quad D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} = 8 + 2 \operatorname{Re} \{c_1^2 c_2\} - 2 |c_2|^2 - 4 |c_1|^2 \geq 0$$

$$\Leftrightarrow 2c_2 = c_1^2 + x(4 - c_1^2),$$

for some  $x$ ,  $|x| \leq 1$ . For  $n = 3$ ,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_1 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} \geq 0$$

and is equivalent to

$$(2.3) \quad |(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2.$$

Simplifying the expressions (2.2) and (2.3), we get

$$(2.4) \quad 4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,$$

with  $|z| \leq 1$ . In obtaining our results, we refer to the classical method devised by Libera and Złotkiewicz [7] and used by several authors in the literature.

### 3. Main results.

**Theorem 3.1.** *If  $f(z) \in ST_s$  then  $|a_2a_3 - a_4| \leq \frac{1}{2}$ .*

**Proof.** For the function  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in ST_s$ , by virtue of Definition 1.1, there exists an analytic function  $p \in \mathcal{P}$  in the unit disc  $E$  with  $p(0) = 1$  and  $\operatorname{Re} p(z) > 0$  such that

$$(3.1) \quad \frac{2zf'(z)}{f(z) - f(-z)} = p(z) \Leftrightarrow 2zf'(z) = [f(z) - f(-z)]p(z).$$

Replacing  $f(z)$ ,  $f'(z)$ ,  $f(-z)$  and  $p(z)$  with their equivalent series expressions in (3.1), we have

$$2z \left\{ 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \right\} = \left[ \left\{ z + \sum_{n=2}^{\infty} a_n z^n \right\} - \left\{ -z + \sum_{n=2}^{\infty} a_n (-z)^n \right\} \right] \times \left\{ 1 + \sum_{n=1}^{\infty} c_n z^n \right\}.$$

Upon simplification, we obtain

$$(3.2) \quad \begin{aligned} & 1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + 5a_5z^4 \dots \\ & = 1 + c_1z + (c_2 + a_3)z^2 + (c_3 + c_1a_3)z^3 + (c_4 + c_2a_3 + a_5)z^4 + \dots \end{aligned}$$

Equating the coefficients of like powers of  $z$ ,  $z^2$ ,  $z^3$  and  $z^4$  respectively in (3.2), after simplifying, we get

$$(3.3) \quad a_2 = \frac{c_1}{2}; \quad a_3 = \frac{c_2}{2}; \quad a_4 = \frac{1}{8}(2c_3 + c_1c_2); \quad a_5 = \frac{1}{8}(2c_4 + c_2^2).$$

Substituting the values of  $a_2$ ,  $a_3$  and  $a_4$  from (3.3) in the functional  $|a_2a_3 - a_4|$  for the function  $f \in ST_s$ , we obtain

$$(3.4) \quad |a_2a_3 - a_4| = \frac{1}{8}|c_1c_2 - 2c_3|.$$

From Lemma 2.2, substituting the values of  $c_2$  and  $c_3$  from (2.2) and (2.4) respectively, on the right-hand side of the expression (3.4), we have

$$\begin{aligned} |c_1c_2 - 2c_3| &= \left| c_1 \frac{1}{2} \{c_1^2 + x(4 - c_1^2)\} - 2 \cdot \frac{1}{4} \{c_1^3 + 2c_1(4 - c_1^2)x \right. \\ &\quad \left. - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\} \right|. \end{aligned}$$

Using the facts  $|z| < 1$  and  $|pa + qb| \leq |p||a| + |q||b|$ , where  $p$ ,  $q$ ,  $a$  and  $b$  are real numbers, after simplifying, we get

$$(3.5) \quad 2|c_1c_2 - 2c_3| \leq |2(4 - c_1^2) + c_1(4 - c_1^2)|x| + (c_1 + 2)(4 - c_1^2)|x|^2.$$

Since  $c_1 = c \in [0, 2]$ , noting that  $c_1 + a \geq c_1 - a$  where  $a \geq 0$ , applying triangle inequality and replacing  $|x|$  by  $\mu$  on the right hand side of the above inequality, we have

$$(3.6) \quad 2|c_1c_2 - 2c_3| \leq \{2 + c\mu + (c - 2)\mu^2\} (4 - c^2) = F(c, \mu),$$

for  $0 \leq \mu = |x| \leq 1$ , where

$$(3.7) \quad F(c, \mu) = \{2 + c\mu + (c - 2)\mu^2\} (4 - c^2).$$

Now, we maximize the function  $F(c, \mu)$  on the closed region  $[0, 2] \times [0, 1]$ . From (3.7), we get

$$(3.8) \quad \frac{\partial F}{\partial \mu} = \{c + 2(c - 2)\mu\} (4 - c^2)$$

and

$$(3.9) \quad \frac{\partial F}{\partial c} = \{\mu + \mu^2\} (4 - c^2).$$

The only stationary point for the function  $F(c, \mu)$  in the region  $[0, 2] \times [0, 1]$  for which  $\frac{\partial F}{\partial c} = 0$  and  $\frac{\partial F}{\partial \mu} = 0$  simultaneously is  $(0, 0)$ , from the elementary

calculus, we observe that the function  $F(c, \mu)$  attains maximum value at this point only and from (3.7), it is given by

$$(3.10) \quad G_{\max} = F(0, 0) = 8.$$

Simplifying the expressions (3.4) and (3.6) together with (3.10), we obtain

$$|a_2 a_3 - a_4| \leq \frac{1}{2}.$$

This completes the proof of our Theorem 3.1.  $\square$

**Theorem 3.2.** *If  $f(z) \in ST_s$ , then  $|a_3 - a_2^2| \leq 1$  and the inequality is sharp for the values  $c_1 = c = 0$ ,  $c_2 = 2$  and  $x = 1$ .*

**Proof.** Substituting the values  $a_2$  and  $a_3$  from (3.3) into the functional  $|a_3 - a_2^2|$ , we obtain

$$(3.11) \quad 4|a_3 - a_2^2| = |2c_2 - c_1^2|.$$

Substituting the value of  $c_2$  from (2.2) of Lemma 2.2 on the right-hand side of (3.11), we get

$$(3.12) \quad |2c_2 - c_1^2| = |(4 - c_1^2)x|.$$

Since  $c_1 = c \in [0, 2]$ , replacing  $|x|$  by  $\mu$  on the right hand side of the above expression, we see that

$$(3.13) \quad |2c_2 - c_1^2| \leq (4 - c^2)\mu = F(c, \mu),$$

for  $0 \leq \mu = |x| \leq 1$ . Next, we maximize the function  $F(c, \mu)$  on the closed region  $[0, 2] \times [0, 1]$ . Differentiating  $F(c, \mu)$  in (3.13) partially with respect to  $\mu$ , we obtain

$$(3.14) \quad \frac{\partial F}{\partial \mu} = (4 - c^2).$$

From (3.14), we observe that  $\frac{\partial F}{\partial \mu} > 0$ , for  $0 < \mu < 1$  and  $0 < c < 2$ . Therefore,  $F(c, \mu)$  is an increasing function of  $\mu$  and hence it cannot have maximum value at any point in the interior of the closed region  $[0, 2] \times [0, 1]$ .

Moreover, for fixed  $c \in [0, 2]$ , we have

$$(3.15) \quad \max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c) = (4 - c^2),$$

$$(3.16) \quad G'(c) = -2c.$$

From the expression (3.16), we observe that  $G'(c) \leq 0$  for every  $c \in [0, 2]$ . Therefore,  $G(c)$  becomes a decreasing function of  $c$ , whose maximum value occurs at  $c = 0$  only, from (3.15), it is given by

$$(3.17) \quad G_{\max} = G(0) = 4.$$

Simplifying the expressions (3.11), (3.13) along with (3.17), we obtain

$$(3.18) \quad |a_3 - a_2^2| \leq 1.$$

This completes the proof of our Theorem 3.2.  $\square$

**Theorem 3.3.** *If  $f(z) \in ST_s$ , then  $|a_k| \leq 1$ , for  $k \in \{2, 3, 4, \dots\}$  and the inequality is sharp.*

**Proof.** Using the fact that  $|c_n| \leq 2$ , for  $n \in N = \{1, 2, 3, \dots\}$ , with the help of  $c_2$  and  $c_3$  values given in (2.2) and (2.4) respectively, together with the values determined in (3.3), we obtain  $|a_k| \leq 1$ , for  $k \in \{2, 3, 4, \dots\}$ . This completes the proof of our Theorem 3.3.  $\square$

Substituting the results of Theorems 3.1, 3.2, 3.3 and the inequality  $|a_2a_4 - a_3^2| \leq 1$  (see [13]) in the inequality (1.4), we obtain the following corollary.

**Corollary 3.4.** *Let  $f(z) \in ST_s$  then  $|H_3(1)| \leq \frac{5}{2}$ .*

**Theorem 3.5.** *If  $f(z) \in CV_s$  then  $|a_2a_3 - a_4| \leq \frac{4}{27}$ .*

**Proof.** Let  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CV_s$ , from the Definition 1.2, there exists an analytic function  $p \in \mathcal{P}$  in the unit disc  $E$  with  $p(0) = 1$  and  $\operatorname{Re} p(z) > 0$  such that

$$(3.19) \quad \frac{2\{zf'(z)\}'}{f'(z) + f'(-z)} = p(z) \Leftrightarrow 2\{zf'(z)\}' = \{f'(z) + f'(-z)\}p(z).$$

Replacing  $f'(z)$ ,  $f''(z)$ ,  $f'(-z)$  and  $p(z)$  with their series equivalent expressions in (3.20) and applying the same procedure as described in Theorem 3.1, we get

$$(3.20) \quad a_2 = \frac{c_1}{4}; \quad a_3 = \frac{c_2}{6}; \quad a_4 = \frac{1}{32}(2c_3 + c_1c_2); \quad a_5 = \frac{1}{40}(2c_4 + c_2^2).$$

Substituting the values of  $a_2, a_3$ , and  $a_4$  from (3.21) in  $|a_2a_3 - a_4|$  for the function  $f \in CV_s$ , upon simplification, we obtain

$$(3.21) \quad |a_2a_3 - a_4| = \frac{1}{96}|c_1c_2 - 6c_3|.$$

Applying the same procedure as described in Theorem 3.1, we arrive at

$$(3.22) \quad 2|c_1c_2 - 6c_3| \leq [2c^3 + \{6 + 5c\mu + 3(c-2)\mu^2\}(4-c^2)] = F(\mu),$$

for  $0 \leq \mu \leq 1$ , where

$$(3.23) \quad F(\mu) = 2c^3 + \{6 + 5c\mu + 3(c-2)\mu^2\}(4-c^2).$$

Next, we maximize the function  $F(\mu)$  on the closed region  $[0, 2] \times [0, 1]$ . Note that  $F'(\mu) \geq F'(1) > 0$ . Then there exists  $c^* \in [0, 2]$  such that  $F'(\mu) > 0$  for  $c \in [c^*, 2]$  and  $F'(\mu) \leq 0$  otherwise. Then for  $c \in [c^*, 2]$ ,  $F(\mu) \leq F(1)$ , that is

$$(3.24) \quad 2|c_1c_2 - 6c_3| \leq -6c^3 + 32c = G(c),$$

where

$$(3.25) \quad G(c) = -6c^3 + 32c,$$

$$(3.26) \quad G'(c) = -18c^2 + 32.$$

For optimum value of  $G(c)$ , consider  $G'(c) = 0$ . From the equation (3.26), we obtain  $c = \pm \frac{4}{3}$ . Since  $c \in [0, 2]$ , consider  $c = \frac{4}{3}$  ( $c^*$ ) only. Further, we observe that  $F(c, \mu)$  attains the maximum value at the point  $[\frac{4}{3}, 1]$  only and from (3.25) it is given by

$$(3.27) \quad G_{\max} = \frac{256}{9}.$$

Simplifying the expressions (3.21), (3.24) along with (3.27), we obtain

$$(3.28) \quad |a_2a_3 - a_4| \leq \frac{4}{27}.$$

This completes the proof of our Theorem 3.5.  $\square$

The following results are straightforward verification on applying the same procedure of Theorems 3.2 and 3.3 respectively.

**Theorem 3.6.** *If  $f(z) \in CV_s$ , then  $|a_3 - a_2^2| \leq \frac{1}{3}$  and the inequality is sharp for the values  $c_1 = c = 0$ ,  $c_2 = 2$  and  $x = 1$ .*

**Theorem 3.7.** *If  $f(z) \in CV_s$ , then  $|a_k| \leq \frac{1}{k}$ , for  $k \in \{2, 3, 4, \dots\}$ .*

For  $f(z) \in CV_s$ , using the result  $|a_2a_4 - a_3^2| \leq \frac{1}{9}$  (see [13]) along with the results of Theorems 3.5, 3.6, 3.7 in the inequality (1.4), we have the following corollary.

**Corollary 3.8.** *If  $f(z) \in CV_s$  then  $|H_3(1)| \leq \frac{19}{135}$ .*

**Acknowledgement.** The authors express sincere thanks to the esteemed Referee(s) for their careful readings, comments and valuable suggestions, which helped them to improve the presentation of the paper.

#### REFERENCES

- [1] Ali, R. M., *Coefficients of the inverse of strongly starlike functions*, Bull. Malays. Math. Sci. Soc. (second series) **26**(1) (2003), 63–71.
- [2] Babalola, K. O., *On  $H_3(1)$  Hankel determinant for some classes of univalent functions*, Inequality Theory and Applications **6** (2010), 1–7.
- [3] Das, R. N., Singh, P., *On subclass of schlicht mappings*, Indian J. Pure and Appl. Math. **8** (1977), 864–872.
- [4] Duren, P. L., *Univalent Functions*, Springer, New York, 1983.
- [5] Grenander, U., Szegő, G., *Toeplitz Forms and Their Applications*, 2nd ed., Chelsea Publishing Co., New York, 1984.
- [6] Janteng, A., Halim, S. A., Darus, M., *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal. (Ruse) **1**(13) (2007), 619–625.
- [7] Libera, R. J., Zlotkiewicz, E. J., *Coefficient bounds for the inverse of a function with derivative in  $\mathcal{P}$* , Proc. Amer. Math. Soc. **87** (1983), 251–257.



- [8] Pommerenke, Ch., *Univalent Functions*, Vandenhoeck and Ruprecht, Gottingen, 1975.
- [9] Pommerenke, Ch., *On the coefficients and Hankel determinants of univalent functions*, J. Lond. Math. Soc. **41** (1966), 111–122.
- [10] Prithvipal Singh, *A study of some subclasses of analytic functions in the unit disc*, Ph.D. Thesis (1979), I.I.T. Kanpur.
- [11] Raja, M., Malik, S. N., *Upper bound of third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli*, J. Inq. Appl. (2013), vol. 2013.
- [12] RamReddy, T., *A study of certain subclasses of univalent analytic functions*, Ph.D. Thesis (1983), I.I.T. Kanpur.
- [13] RamReddy, T., Vamshee Krishna, D., *Hankel determinant for starlike and convex functions with respect to symmetric points*, J. Ind. Math. Soc. (N. S.) **79** (1–4) (2012), 161–171.
- [14] Ratanchand, *Some aspects of functions analytic in the unit disc*, Ph.D. Thesis (1978), I.I.T. Kanpur.
- [15] Sakaguchi, K., *On a certain univalent mapping*, J. Math. Soc. Japan **11** (1959), 72–75.
- [16] Simon, B., *Orthogonal Polynomials on the Unit Circle, Part 1. Classical Theory*, American Mathematical Society, Providence (RI), 2005.
- [17] Vamshee Krishna, D., Venkateswarlu, B., RamReddy, T., *Third Hankel determinant for certain subclass of  $p$ -valent functions*, Complex Var. and Elliptic Eqns. **60** (9) (2015), 1301–1307.
- [18] Vamshee Krishna, D., RamReddy, T., *Coefficient inequality for certain  $p$ -valent analytic functions*, Rocky Mountain J. Math. **44** (6) (2014), 1941–1959.

D. Vamshee Krishna  
Department of Mathematics  
GIT, GITAM University  
Visakhapatnam-530 045, A.P.  
India  
e-mail: vamsheekrishna1972@gmail.com

B. Venkateswarlu  
Department of Mathematics  
GIT, GITAM University  
Visakhapatnam-530 045, A.P.  
India  
e-mail: bvlmaths@gmail.com

T. RamReddy  
Department of Mathematics  
Kakatiya University  
Warangal-506 009, T.S.  
India  
e-mail: reddytr2@gmail.com

Received March 27, 2015