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The vertical prolongation of the projectable connections

ABSTRACT. We prove that any first order $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator transforming projectable general connections on an (m_1, m_2, n_1, n_2)-dimensional fibred-fibred manifold $p = (p, \underline{p})\colon (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ into general connections on the vertical prolongation $VY \to M$ of $p\colon Y \to M$ is the restriction of the (rather well-known) vertical prolongation operator $\mathcal{V}$ lifting general connections $\overline{\Gamma}$ on a fibred manifold $Y \to M$ into $\mathcal{V}\overline{\Gamma}$ (the vertical prolongation of $\overline{\Gamma}$) on $VY \to M$.

The aim of this paper is to describe all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators transforming projectable general connections on an (m_1, m_2, n_1, n_2)-dimensional fibred-fibred manifolds into general connections on the vertical prolongation $VY \to M$ of $p\colon Y \to M$. The similar problem for the case of fibred manifolds was solving in [7]. In the paper [1], authors described natural operators transforming connections on fibred manifolds $Y \to M$ into connections on $VY \to M$.

A fibred-fibred manifold is a fibred surjective submersion

$$p = (p, \underline{p})\colon (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$$

between two fibred manifolds $p_Y : Y \to \underline{Y}$ and $p_M : M \to \underline{M}$ covering $\underline{p}\colon \underline{Y} \to \underline{M}$ such that the restrictions of p to the fibres are submersions. Equivalently, the fibred-fibred manifold is a fibred square $p = (p, p_Y, p_M, \underline{p})$, i.e. a commutative square diagram with arrows being surjective submersions

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$p\colon Y\to M$, $p_Y\colon Y\to \underline{Y}$, $p_M\colon M\to \underline{M}$ and $\underline{p}\colon \underline{Y}\to\underline{M}$ such that the system $(p,p_Y)\colon Y\to M\times_{\underline{M}}\underline{Y}$ of maps p and p_Y is a submersion, [2], [6].

If $p^1=(p^1,\underline{p}^1)\colon (p^1_{Y^1}\colon Y^1\to\underline{Y}^1)\to(p^1_{M^1}\colon M^1\to\underline{M}^1)$ is another fibred-fibred manifold, then a fibred-fibred map $f\colon Y\to Y^1$ is the system $f=(f,f_1,f_2,\underline{f})$ of four maps $f\colon Y\to Y^1$, $f_1\colon\underline{Y}\to\underline{Y}^1$, $f_2\colon M\to M^1$ and $\underline{f}\colon\underline{M}\to\underline{M}^1$ such that the relevant cubic diagram is commutative.

A fibred-fibred manifold $p=(p,\underline{p})\colon(p_Y\colon Y\to\underline{Y})\to(p_M\colon M\to\underline{M})$ is of the dimension (m_1,m_2,n_1,n_2) if $\dim Y=m_1+m_2+n_1+n_2$, $\dim M=m_1+m_2$, $\dim\underline{Y}=m_1+n_1$ and $\dim\underline{M}=m_1$. All fibred-fibred manifolds of the dimension (m_1,m_2,n_1,n_2) and their all local fibred-fibred diffeomorphisms form the local admissible category over manifolds in the sense of [3], which we denote by $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$. Any $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object is locally isomorphic to the trivial fibred square $\mathbb{R}^{m_1,m_2,n_1,n_2}$ with vertices $\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, $\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}$, $\mathbb{R}^{m_1}\times\mathbb{R}^{n_1}$, $\mathbb{R}^{m_1}$ and arrows being obvious projections.

The vertical functor $V\colon\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}\to\mathcal{FM}$ (on (m_1,m_2,n_1,n_2)-dimensional fibred-fibred manifolds) is the usual vertical functor $V\colon\mathcal{FM}\to\mathcal{FM}$ on fibred manifolds, where $\mathcal{FM}$ is the category of all fibred manifolds and their morphisms. More precisely, $V\colon\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}\to\mathcal{FM}$ is the functor assigning to any (m_1,m_2,n_1,n_2)-dimensional fibred-fibred manifold $p=(p,\underline{p})\colon(p_Y\colon Y\to\underline{Y})\to(p_M\colon M\to\underline{M})$ the vertical bundle $VY\to Y$ of the corresponding fibred manifold $p\colon Y\to M$ and to any $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map $f=(f,f_1,f_2,\underline{f})$ between (m_1,m_2,n_1,n_2)-dimensional fibred-fibred manifolds $p=(p,\underline{p})\colon(p_Y\colon Y\to\underline{Y})\to(p_M\colon M\to\underline{M})$ and $p^1=(p^1,\underline{p}^1)\colon(p^1_{Y^1}\colon Y^1\to\underline{Y}^1)\to(p^1_{M^1}\colon M^1\to\underline{M}^1)$ the vertical prolongation $Vf\colon VY\to VY^1$ of the corresponding fibred map $f=(f,f_2)$ between corresponding fibred manifolds $p\colon Y\to M$ and $p^1\colon Y^1\to M^1$. Obviously, $V\colon\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}\to\mathcal{FM}$ is a bundle functor in the sense of [3].

A projectable general connection on a fibred-fibred manifold $p=(p,\underline{p})\colon(p_Y\colon Y\to\underline{Y})\to(p_M\colon M\to\underline{M})$ is a pair $\Gamma=(\overline{\Gamma},\underline{\Gamma})$ of general connections $\overline{\Gamma}\colon Y\times_M TM\to TY$ and $\underline{\Gamma}\colon\underline{Y}\times_{\underline{M}}T\underline{M}\to T\underline{Y}$ on fibred manifolds $p\colon Y\to M$ and $\underline{p}\colon\underline{Y}\to\underline{M}$, respectively, such that $Tp_Y\circ\overline{\Gamma}=\underline{\Gamma}\circ(p_Y\times_{p_M}Tp_M)$, [3], [5].

The vertical prolongation $\mathcal{V}\Gamma$ of a projectable general connection $\Gamma=(\overline{\Gamma},\underline{\Gamma})$ on an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object $p=(p,\underline{p})\colon(p_Y\colon Y\to\underline{Y})\to(p_M\colon M\to\underline{M})$ is the vertical prolongation $\mathcal{V}\overline{\Gamma}$ on $VY\to M$ of the corresponding general connection $\overline{\Gamma}$ on the corresponding fibred manifold $p\colon Y\to M$. (The vertical prolongation of a general connection $\overline{\Gamma}$ on a fibred manifold $Y\to M$ is the general connection $\mathcal{V}\overline{\Gamma}$ on $VY\to M$ as in Section 31.1 in [3]. The vertical prolongation of connections on fibred manifolds was also described in [4].)

The general concept of natural operator can be found in [3]. In particular, an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator D transforming projectable general connections $\Gamma = (\overline{\Gamma}, \underline{\Gamma})$ on an (m_1, m_2, n_1, n_2)-dimensional fibred-fibred manifold $p = (p, \underline{p})\colon (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ into general connections $D(\Gamma)$ on $VY \to M$ is a family of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariant regular operators

$$D\colon Con_{proj}(Y \to M) \to Con(VY \to M)$$

for all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-objects $p = (p, \underline{p})\colon (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$, where $Con_{proj}(Y \to M)$ is the set of all projectable general connections on $p = (p, \underline{p})$ and $Con(VY \to M)$ is the set of all general connections on $VY \to M$. The $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariance means that if $\Gamma \in Con_{proj}(Y \to M)$ and $\Gamma^1 \in Con_{proj}(Y^1 \to M^1)$ are f-related by $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map $f\colon Y \to Y^1$ (i.e. $Tf \circ \Gamma = \Gamma^1 \circ (f \times Tf_2)$), then $D(\Gamma)$ and $D(\Gamma^1)$ are Vf-related (i.e. $TVf \circ D(\Gamma) = D(\Gamma^1) \circ (f \times Tf_2)$). The regularity for D means that D transforms smoothly parametrized families of connections into smoothly parametrized families of connections.

Thus (by the canonical character of the vertical prolongation of projectable general connections) the family $\mathcal{V}$ of operators

$$\mathcal{V}\colon Con_{proj}(Y \to M) \to Con(VY \to M), \qquad \mathcal{V}(\Gamma) := \mathcal{V}\overline{\Gamma}$$

for all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-objects $p = (p, \underline{p})\colon (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ is an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator.

One can verify that $\mathcal{V}$ is the first order operator (it means that if $\Gamma, \Gamma^1 \in Con_{proj}(Y \to M)$ have the same 1-jets $j^1_x(\Gamma) = j^1_x(\Gamma^1)$ at $x \in M$, then it holds $\mathcal{V}(\Gamma) = \mathcal{V}(\Gamma^1)$ over x).

Theorem 1. *The operator $\mathcal{V}$ is the unique first order $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator transforming projectable general connections on (m_1, m_2, n_1, n_2)-dimensional fibred-fibred manifolds $p = (p, \underline{p})\colon (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ into general connections on $VY \to M$.*

For $j = 1, \dots, m_2$, $s = 1, \dots, n_2$ we put $[j] := m_1 + j$ and $< s > := n_1 + s$.

Let $x^i, x^{[j]}, y^q, y^{<s>}$ be the usual coordinates on the trivial fibred square $\mathbb{R}^{m_1,m_2,n_1,n_2}$.

Lemma 1. *Let $\Gamma = (\overline{\Gamma}, \underline{\Gamma})$ be a projectable general connection on an (m_1, m_2, n_1, n_2)-dimensional fibred-fibred manifold $p = (p, \underline{p})\colon (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ and let $y_0 \in Y$ be a point. Then there exists an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-chart ψ on Y satisfying conditions $\psi(y_0) = (0,0,0,0)$ and $j^1_{(0,0,0,0)}\psi_*\Gamma = j^1_{(0,0,0,0)}\widetilde{\Gamma}$, where*

(1)
$$
\begin{aligned}
\widetilde{\Gamma} = & \sum_{i=1}^{m_1} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{j=1}^{m_1} dx^{[j]} \otimes \frac{\partial}{\partial x^{[j]}} \\
& + \sum_{i_1,i_2=1}^{m_1} \sum_{q=1}^{n_1} A^q_{i_1 i_2} x^{i_1} dx^{i_2} \otimes \frac{\partial}{\partial y^q} + \sum_{i_1,i_2=1}^{m_1} \sum_{s=1}^{n_2} B^s_{i_1 i_2} x^{i_1} dx^{i_2} \otimes \frac{\partial}{\partial y^{<s>}} \\
& + \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \sum_{s=1}^{n_2} C^s_{ij} x^i dx^{[j]} \otimes \frac{\partial}{\partial y^{<s>}} + \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \sum_{s=1}^{n_2} D^s_{ji} x^{[j]} dx^i \otimes \frac{\partial}{\partial y^{<s>}} \\
& + \sum_{j_1,j_2=1}^{m_2} \sum_{s=1}^{n_2} E^s_{j_1 j_2} x^{[j_1]} dx^{[j_2]} \otimes \frac{\partial}{\partial y^{<s>}}
\end{aligned}
$$

for some real numbers $A^q_{i_1 i_2}$, $B^s_{i_1 i_2}$, C^s_{ij}, D^s_{ji} *and* $E^s_{j_1 j_2}$ *satisfying*

(2) $$A^q_{i_1 i_2} = -A^q_{i_2 i_1},\ B^s_{i_1 i_2} = -B^s_{i_2 i_1},\ C^s_{ij} = -D^s_{ji},\ E^s_{j_1 j_2} = -E^s_{j_2 j_1}$$

for $i, i_1, i_2 = 1, \dots, m_1,\ j, j_1, j_2 = 1, \dots, m_2,\ q = 1, \dots, n_1,\ s = 1, \dots, n_2.$

Proof. Choose a projectable torsion-free classical linear connection ∇ on $p_M \colon M \to \underline{M}$, i.e. a torsion-free classical linear connection ∇ on M such that there exists a unique classical linear connection $\underline{\nabla}$ on $\underline{M}$ which is p_M-related with ∇. By Lemma 4.2 [5], there exists an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-chart ψ on Y covering a ∇-normal fiber coordinate system on M with the center $x_0 = p(y_0)$ such that $\psi(y_0) = (0,0,0,0)$ and such that $j^1_{(0,0,0,0)}(\psi_* \Gamma) = j^1_{(0,0,0,0)} \widetilde{\Gamma}$, where $\widetilde{\Gamma}$ means (1) for some real numbers: $A^q_{i_1 i_2}$, $B^s_{i_1 i_2}$, C^s_{ij}, D^s_{ji} and $E^s_{j_1 j_2}$ satisfying the condition (2) for $i, i_1, i_2 = 1, \dots, m_1,\ j, j_1, j_2 = 1, \dots, m_2$, $q = 1, \dots, n_1,\ s = 1, \dots, n_2$. □

Using Lemma 1, one can prove Theorem 1 as follows.

Proof. Let D be an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator of the first order. Put $\nabla(\Gamma) := D(\Gamma) - \mathcal{V}(\Gamma) \colon VY \to T^*M \otimes V(VY)$. It is sufficient to prove that it holds $\nabla(\Gamma) = 0$. Because of Lemma 1, the first order of Δ and invariance of ∇ with respect to charts of the fibred-fibred manifold, it is sufficient to prove that $\langle \Delta(\Gamma)_{|v}, u \rangle = 0$ for any $u \in T_{(0,0)}(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2})$, any $v \in (V\mathbb{R}^{m_1,m_2,n_1,n_2})_{(0,0,0,0)}$ and any projectable general connection Γ on $\mathbb{R}^{m_1,m_2,n_1,n_2}$ of the form (1) for any real numbers $A^q_{i_1 i_2}$, $B^s_{i_1 i_2}$, C^s_{ij}, D^s_{ji} and $E^s_{j_1 j_2}$ satisfying (2) for $i, i_1, i_2 = 1, \dots, m_1,\ j, j_1, j_2 = 1, \dots, m_2,\ q = 1, \dots, n_1,\ s = 1, \dots, n_2$. Using the invariance of Δ with respect to the base homotheties $(tx^i, tx^{[j]}, y^q, y^{<s>})$ for $t > 0$, we obtain the condition of homogeneity of the form

$$\langle \Delta(\Gamma^t)_{|v}, u \rangle = t \langle \Delta(\Gamma)_{|v}, u \rangle,$$

where Γ^t means Γ with the coefficients $t^2 A^q_{i_1 i_2}$, $t^2 B^s_{i_1 i_2}$, $t^2 C^s_{ij}$, $t^2 D^s_{ji}$, $t^2 E^s_{j_1 j_2}$ instead of $A^q_{i_1 i_2}$, $B^s_{i_1 i_2}$, C^s_{ij}, D^s_{ji},$E^s_{j_1 j_2}$.

By the homogeneous function theorem this type of homogeneity yields directly that $\langle \Delta(\Gamma)_{|v}, u\rangle = 0$. $\square$

In case of $m_1 = m$, $n_1 = n$, $m_2 = 0$, $n_2 = 0$ we have $\mathcal{F}^2\mathcal{M}_{m,0,n,0} = \mathcal{F}\mathcal{M}_{m,n}$, the category of the (m,n)-dimensional fibred manifolds and their local fibre diffeomorphisms. In this case, the projectable general connections are the general connections.

Corollary 1. *The operator $\mathcal{V}$ is a unique $\mathcal{F}\mathcal{M}_{m,n}$-natural operator of the first order transforming the general connections on an (m,n)-dimensional fibred manifold $p\colon Y \to M$ into the general connections on $VY \to M$.*

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