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Classes of meromorphic multivalent functions with Montel's normalization

Abstract. In the paper we define classes of meromorphic multivalent functions with Montel's normalization. We investigate the coefficients estimates, distortion properties, the radius of starlikeness, subordination theorems and partial sums for the defined classes of functions. Some remarks depicting consequences of the main results are also mentioned.

1. Introduction. Let $\mathcal{M}$ denote the class of functions which are *analytic* in $\mathcal{D} = \mathcal{D}(1)$, where

$$\mathcal{D}(r) = \{z \in \mathbb{C} : 0 < |z| < r\},$$

and let $\mathcal{M}\,(p,k)$ $(p,k \in \mathbb{N} := \{1,2,3\dots\})$ denote the class of meromorphic functions $f \in \mathcal{M}$ of the form

$$f(z) = a_{-p}z^{-p} + \sum_{n=k}^{\infty} a_n z^n \quad (z \in \mathcal{D};\ a_{-p} > 0). \tag{1}$$

For a meromorphic multivalent function $f \in \mathcal{M}\,(p,k)$ the normalization $z^p f\,(z)|_{z=0} = 1$ is classical. Then we have

$$f(z) = z^{-p} + \sum_{n=k}^{\infty} a_n z^n \quad (z \in \mathcal{D};\ a_{-p} > 0). \tag{2}$$

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One can obtain interesting results by applying normalization related to the Montel's normalization (*cf.* [7]) of the form

$$z^p f(z)|_{z=\rho e^{i\eta}} = 1, \tag{3}$$

where ρ is a fixed real number, $-1 < \rho < 1$.

We denote by $\mathcal{M}_\rho^\eta(p,k)$ the classes of functions $f \in \mathcal{M}(p,k)$ with Montel's normalization (3).

Also, by $\mathcal{T}^\eta(p,k)$ $(\eta \in \mathbb{R})$ we denote the class of functions $f \in \mathcal{M}(p,k)$ of the form

$$f(z) = a_{-p} z^{-p} + \sum_{n=k}^{\infty} |a_n|\, e^{-(n+p)\eta} z^n \quad (z \in \mathcal{D}). \tag{4}$$

For $\eta = 0$ we obtain the class $\mathcal{T}^0(p,k)$ of functions with positive coefficients.

Finally, motivated by Silverman [10], we define the class

$$\mathcal{T}(p,k) := \bigcup_{\eta \in \mathbb{R}} \mathcal{T}^\eta(p,k), \tag{5}$$

which is called the class of functions with varying argument of coefficients.

Let $\alpha \in \langle 0, p)$, $r \in (0,1\rangle$. A function $f \in \mathcal{M}(p,k)$ is said to be *starlike of order* α in $\mathcal{D}(r)$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in \mathcal{D}(r)). \tag{6}$$

We denote by $\mathcal{MS}_p^*(\alpha)$ the class of all functions $f \in \mathcal{M}(p, p+1)$, which are starlike of order α in $\mathcal{D}$.

It is easy to show that for a function f from the class $\mathcal{T}(p,k)$ the condition (6) is equivalent to the following

$$\left|\frac{zf'(z)}{f(z)} + p\right| < p - \alpha \quad (z \in \mathcal{D}(r)). \tag{7}$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{M}(p,k)$. We define *the radius of starlikeness of order* α for the class $\mathcal{B}$ by

$$R_\alpha^*(\mathcal{B}) = \inf_{f \in \mathcal{B}} \left(\sup\{r \in (0,1] : f \text{ is starlike of order } \alpha \text{ in } \mathcal{D}(r)\}\right).$$

Let functions f, F be analytic in $\mathcal{U}$. We say that the function f is *subordinate* to the function F, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if and only if there exists a function ω analytic in $\mathcal{U}$, $|\omega(z)| \le |z|$ $(z \in \mathcal{U})$, such that

$$f(z) = F(\omega(z)) \quad (z \in \mathcal{U}).$$

In particular, if F is univalent in $\mathcal{U}$, we have the following equivalence

$$f(z) \prec F(z) \Longleftrightarrow [f(0) = F(0) \wedge f(\mathcal{U}) \subset F(\mathcal{U})].$$

For functions $f, g \in \mathcal{M}$ of the form

$$f(z) = \sum_{n=-p}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (z \in \mathcal{D}),$$

by $f*g$ we denote *the Hadamard product* (or *convolution* of f and g), defined by

$$(f*g)(z) = \sum_{n=-p}^{\infty} a_n b_n z^n \quad (z \in \mathcal{D}).$$

Let A, B be real parameters, $0 \le B \le 1$, $-1 \le A < B$, and let $\varphi, \phi \in \mathcal{M}(p,k)$.

By $\mathcal{W}(p,k;\phi,\varphi;A,B)$ we denote the class of functions $f \in \mathcal{M}(p,k)$ such that

$$(\varphi * f)(z) \neq 0 \quad (z \in \mathcal{D}) \tag{8}$$

and

$$\frac{(\phi * f)(z)}{(\varphi * f)(z)} \prec \frac{1+Az}{1+Bz}. \tag{9}$$

Now, we define the classes of functions with varying argument of coefficients related to the class $\mathcal{W}(p,k;\phi,\varphi;A,B)$. Let us denote

$$\begin{aligned}
\mathcal{W}_\rho(p,k;\phi,\varphi;A,B) &:= \mathcal{M}_\rho(p,k) \cap \mathcal{W}(p,k;\phi,\varphi;A,B), \\
\mathcal{TW}^\eta(p,k;\phi,\varphi;A,B) &:= \mathcal{T}^\eta(p,k) \cap \mathcal{W}(p,k;\phi,\varphi;A,B), \\
\mathcal{TW}^\eta_\rho(p,k;\phi,\varphi;A,B) &:= \mathcal{M}^\eta_\rho(p,k) \cap \mathcal{TW}^\eta(p,k;\phi,\varphi;A,B), \\
\mathcal{TW}_\rho(p,k;\phi,\varphi;A,B) &:= \mathcal{T}(p,k) \cap \mathcal{W}_\rho(p,k;\phi,\varphi;A,B).
\end{aligned}$$

For the presented investigations we assume that φ, ϕ are functions of the form

$$\varphi(z) = z^{-p} + \sum_{n=k}^{\infty} \alpha_n z^n, \quad \phi(z) = z^{-p} - \sum_{n=k}^{\infty} \beta_n z^n \quad (z \in \mathcal{D}), \tag{10}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are nonnegative real, and $\alpha_n + \beta_n > 0$ $(n = k, k+1, \dots)$. Moreover, let us put

$$d_n := (1+B)\beta_n + (A+1)\alpha_n \quad (n = k, k+1, \dots). \tag{11}$$

The family $\mathcal{W}(p,k;\phi,\varphi;A,B)$ unifies a lot of new and also well-known classes of meromorphic functions. We list a few of them in the last section.

The object of the present paper is to investigate the coefficients estimates, distortion properties, the radius of starlikeness, subordination theorems and partial sums for the defined classes of functions. Some remarks depicting consequences of the main results are also mentioned.

2. Coefficients estimates. We first mention a sufficient condition for a function to belong to the class $\mathcal{W}(p,k;\phi,\varphi;A,B)$.

Theorem 1. *Let $\{d_n\}$ be defined by* (11), $-1 \le A < B \le 1$. *If a function $f \in \mathcal{M}(p,k)$ of the form* (1) *satisfies the condition*

$$\sum_{n=k}^{\infty} d_n |a_n| \le (B-A)\, a_{-p}, \tag{12}$$

then f belongs to the class $\mathcal{W}(p,k;\phi,\varphi;A,B)$.

Proof. A function f of the form (1) belongs to the class $\mathcal{W}(p,k;\phi,\varphi;A,B)$ if and only if there exists a function ω, $|\omega(z)| \le |z| \quad (z \in \mathcal{D})$, such that

$$\frac{(\phi * f)(z)}{(\varphi * f)(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (z \in \mathcal{D}),$$

or equivalently

$$\left| \frac{z^p (\phi * f)(z) - z^p (\varphi * f)(z)}{Bz^p (\phi * f)(z) - Az^p (\varphi * f)(z)} \right| < 1 \quad (z \in \mathcal{D}). \tag{13}$$

Thus, it is sufficient to prove that

$$|z^p (\phi * f)(z) - z^p (\varphi * f)(z)| - |Bz^p (\phi * f)(z) - Az^p (\varphi * f)(z)| < 0$$

$(z \in \mathcal{D})$. Indeed, letting $|z| = r$ $(0 < r < 1)$, and using (12), we have

$$\begin{aligned}
&|z^p (\phi * f)(z) - z^p (\varphi * f)(z)| - |Bz^p (\phi * f)(z) - Az^p (\varphi * f)(z)| \\
&= \left| \sum_{n=k}^{\infty} (\beta_n + \alpha_n)\, a_n z^{n+p} \right| - \left| (B-A)\, a_{-p} - \sum_{n=k}^{\infty} (B\beta_n + A\alpha_n)\, a_n z^{n+p} \right| \\
&\le \sum_{n=k}^{\infty} (\beta_n + \alpha_n)\, |a_n|\, r^{n+p} - (B-A)\, a_{-p} + \sum_{n=k}^{\infty} (B\beta_n + A\alpha_n)\, |a_n|\, r^{n+p} \\
&\le \sum_{n=k}^{\infty} d_n |a_n|\, r^{n+p} - (B-A)\, a_{-p} < 0,
\end{aligned}$$

hence $f \in \mathcal{W}(p,k;\phi,\varphi;A,B)$. □

Theorem 2. *Let $f \in \mathcal{T}^{\eta}(p,k)$ be a function of the form* (4). *Then f belongs to the class $\mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B)$ if and only if the condition* (12) *holds true.*

Proof. In view of Theorem 1 we need only to show that each function f from the class $\mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B)$ satisfies the coefficient inequality (12). Let $f \in \mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B)$. Then by (13) and (1), we have

$$\left| \frac{\sum\limits_{n=k}^{\infty} (\beta_n + \alpha_n)\, a_n z^{n+p}}{(B-A)\, a_{-p} - \sum\limits_{n=k}^{\infty} (B\beta_n + A\alpha_n)\, a_n z^{n+p}} \right| < 1 \quad (z \in \mathcal{D}).$$

Thus, putting $z = re^{i\eta}$ $(0 \le r < 1)$, and applying (4), we obtain

$$\frac{\sum\limits_{n=k}^{\infty} (\beta_n + \alpha_n) |a_n| r^{n+p}}{(B - A) a_{-p} - \sum\limits_{n=k}^{\infty} (B\beta_n + A\alpha_n) |a_n| r^{n+p}} < 1. \tag{14}$$

It is clear that the denominator of the left hand said can not vanish for $r \in \langle 0, 1)$. Moreover, it is positive for $r = 0$, and in consequence for $r \in \langle 0, 1)$. Thus, by (14) we have

$$\sum_{n=k}^{\infty} [(1 + B) \beta_n + (1 + A) \alpha_n] |a_n| r^{n+p} < (B - A) a_{-p},$$

which, upon letting $r \to 1^-$, readily yields the assertion (12). □

By applying Theorem 2 we can deduce the following result.

Theorem 3. *Let $f \in \mathcal{T}^\eta (p, k)$ be a function of the form (4). Then f belongs to the class $\mathcal{TW}_\rho^\eta (p, k; \phi, \varphi; A, B)$ if and only if it satisfies (3) and*

$$\sum_{n=k}^{\infty} \left(d_n + (B - A) \rho^{n+p}\right) |a_n| \le B - A, \tag{15}$$

where $\{d_n\}$ is defined by (11).

Proof. For a function f of the form (1) with the normalization (3), we have

$$a_{-p} = 1 - \sum_{n=k}^{\infty} |a_n| \rho^{n+p}. \tag{16}$$

Applying the equality (16) to (12), we obtain the assertions (15). □

By applying Theorem 3 we obtain the following lemma.

Lemma 1. *Let $\{d_n\}$ be defined by (11), $-1 < \rho < 1$, and let us assume that there exists an integer n_0 $(n_0 \in \mathbb{N}_k := \{k, k+1, \dots\})$ such that*

$$d_{n_0} + (B - A) \rho^{n_0+p} \le 0. \tag{17}$$

Then the function

$$f_{n_0}(z) = \left(1 + a\rho^{n_0+p}\right) z^{-p} + a\ e^{-i(n_0+p)\eta} z^{n_0} \quad (z \in \mathcal{D})$$

belongs to the class $\mathcal{TW}_\rho^\eta (p, k; \phi, \varphi; A, B)$ for all positive real numbers a. Moreover, for all n $(n \in \mathbb{N}_k)$ such that

$$d_n + (B - A) \rho^{n+p} > 0, \tag{18}$$

the functions

$$f_n(z) = \left(1 + a\rho^{n_0+p} + bz^{n+p}\right) z^{-p} + ae^{-i(n_0+p)\eta} z^{n_0} + be^{-i(n+p)\eta} z^n \quad (z \in \mathcal{D}),$$

where
$$b=\frac{B-A-\left(d_{n_0}+(B-A)\,\rho^{n_0+p}\right)a}{d_n+(B-A)\,\rho^{n+p}},$$
belong to the class $\mathcal{TW}_{\rho}^{\eta}(p,k;\phi,\varphi;A,B)$.

By Lemma 1 and Theorem 3, we have the following theorem.

Theorem 4. *Let a function* f *of the form* (4) *belong to the class* $\mathcal{TW}_{\rho}^{\eta}(p,k;\phi,\varphi;A,B)$ *and let* $\{d_n\}$ *be defined by* (11). *Then all of the coefficients* a_n *for which*
$$d_n+(B-A)\,\rho^{n+p}=0$$
are unbounded. Moreover, if there exists an integer n_0 $(n_0\in\mathbb{N}_k)$ *such that*
$$d_{n_0}+(B-A)\,\rho^{n_0+p}<0,$$
then all of the coefficients of the function f *are unbounded. In the remaining cases*
$$|a_n|\leq\frac{B-A}{d_n+(B-A)\,\rho^{n+p}}\ . \tag{18}$$
The result is sharp, the functions f_n *of the form*
$$f_{n,\eta}(z)=\frac{d_nz^{-p}+(B-A)\,e^{-i(n+p)\eta}z^n}{d_n+(B-A)\,\rho^{n+p}}\quad(z\in\mathcal{D};\ n=k,k+1,\ldots) \tag{19}$$
are the extremal functions.

By putting $\rho=0$ in Theorem 3 and Corollary 4, we have the corollaries listed below.

Corollary 1. *Let* $f\in\mathcal{T}^{\eta}(p,k)$ *be a function of the form* (2). *Then* f *belongs to the class* $\mathcal{TW}_{0}^{\eta}(p,k;\phi,\varphi;A,B)$ *if and only if*
$$\sum_{n=k}^{\infty}d_n\,|a_n|\leq B-A, \tag{19}$$
where $\{d_n\}$ *is defined by* (11).

Corollary 2. *If a function* f *of the form* (2) *belongs to the class* $\mathcal{TW}_{0}^{\eta}(p,k;\phi,\varphi;A,B)$, *then*
$$|a_n|\leq\frac{B-A}{d_n}\quad(n=k,k+1,\ldots), \tag{20}$$
where d_n *is defined by* (11). *The result is sharp. The functions* $f_{n,\eta}$ *of the form*
$$f_{n,\eta}(z)=z^{-p}+\frac{B-A}{d_n}e^{-i(p+n)\eta}z^n\quad(z\in\mathcal{D};\ n=k,k+1,\ldots) \tag{21}$$
are the extremal functions.

3. Distortion theorems. From Theorem 2 we have the following lemma.

Lemma 2. *Let a function f of the form* (1) *belong to the class* $\mathcal{TW}_\rho^\eta(p,k;\phi,\varphi;A,B)$. *If the sequence $\{d_n\}$ defined by* (11) *satisfies the inequality*

$$0 < d_k + (B-A)\,\rho^{k+p} \le d_n + (B-A)\,\rho^{n+p} \qquad (n = k, k+1, \ldots)\,, \tag{22}$$

then

$$\sum_{n=k}^{\infty} |a_n| \le \frac{B-A}{d_k + (B-A)\,\rho^{k+p}}.$$

Moreover, if

$$0 < \frac{d_k + (B-A)\,\rho^{k+p}}{k} \le \frac{d_n + (B-A)\,\rho^{n+p}}{n} \qquad (n = k, k+1, \ldots)\,, \tag{23}$$

then

$$\sum_{n=k}^{\infty} n\,|a_n| \le \frac{k\,(B-A)}{d_k + (B-A)\,\rho^{k+p}}.$$

Remark 1. The second part of Lemma 2 can be rewritten in terms of σ-neighborhood N_σ defined by

$$N_\sigma = \left\{ f(z) = a_{-p} z^{-p} + \sum_{n=k}^{\infty} a_n z^n \in \mathcal{T}^\eta(p,k) : \quad \sum_{n=k}^{\infty} n\,|a_n| \le \sigma \right\}$$

as the following corollary.

Corollary 3. *If the sequence $\{d_n\}$ defined by* (11) *satisfies* (23)*, then*

$$\mathcal{TW}_\rho^\eta(p,k;\phi,\varphi;A,B) \subset N_\sigma,$$

where

$$\delta = \frac{k\,(B-A)}{d_k + (B-A)\,\rho^{k+p}}.$$

Theorem 5. *Let a function f belong to the class $\mathcal{TW}_\rho^\eta(p,k;\phi,\varphi;A,B)$ and let $z \in \mathcal{D}$, $|z| = r$. If the sequence $\{d_n\}$ defined by* (11) *satisfies* (22)*, then*

$$\phi(r) \le |f(z)| \le \frac{d_k r^{-p} + (B-A)\,r^k}{d_k + (B-A)\,\rho^{k+p}}, \tag{24}$$

where

$$\phi(r) := \begin{cases} r^{-p} & (r \le \rho) \\ \frac{d_k r^{-p} + (B-A) r^k}{d_k + (B-A)\rho^{k+p}} & (r > \rho)\,. \end{cases} \tag{25}$$

Moreover, if (23) *holds, then*

$$\begin{aligned}(26)\qquad & pa_{-p}r^{-p-1}-\frac{k\,(B-A)}{d_k+(B-A)\,\rho^{k+p}}r^{k-1}\\ &\qquad\le\left|f'(z)\right|\le\frac{pd_kr^{-p}+k\,(B-A)\,r^{k-1}}{d_k+(B-A)\,\rho^{k+p}}.\end{aligned}$$

The result is sharp, with the extremal function $f_{k,\eta}$ *of the form* (21) *and* $f(z)=z^{-p}$.

Proof. Suppose that the function f of the form (1) belongs to the class $\mathcal{TW}^{\eta}_{\rho}\,(p,k;\phi,\varphi;A,B)$. By Lemma 2 we have

$$\begin{aligned}|f(z)| &= \left|a_{-p}z^{-p}+\sum_{n=k}^{\infty}a_nz^n\right|\le r^{-p}\left(a_{-p}+\sum_{n=k}^{\infty}|a_n|\,r^{n+p}\right)\\ &\le r^{-p}\left(1+\sum_{n=k}^{\infty}|a_n|\,\rho^{n+p}+\sum_{n=k}^{\infty}|a_n|\,r^{n+p}\right)\\ &\le r^{-p}\left(1+(\rho^{k+p}+r^{k+p})\sum_{n=k}^{\infty}|a_n|\right)\le\frac{d_kr^{-p}+(B-A)\,r^k}{d_k+(B-A)\,\rho^{k+p}}\end{aligned}$$

and

$$\begin{aligned}(27)\qquad |f(z)| &\ge r^{-p}\left(a_{-p}-\sum_{n=k}^{\infty}|a_n|\,r^{n+p}\right)\\ &= r^{-p}\left(1+\sum_{n=k}^{\infty}(\rho^{n+p}-r^{n+p})\,|a_n|\right).\end{aligned}$$

If $r\le\rho$, then we obtain $|f(z)|\ge r^{-p}$. If $r>\rho$, then the sequence $\{(\rho^{n+p}-r^{n+p})\}$ is decreasing and negative. Thus, by (27), we obtain

$$|f(z)|\ge r^{-p}\left(1-(r^{k+p}-\rho^{k+p})\sum_{n=2}^{\infty}a_n\right)\ge\frac{d_kr^{-p}+(B-A)\,r^k}{d_k+(B-A)\,\rho^{k+p}},$$

and we have the assertion (24). Making use of Lemma 2, in conjunction with (16), we readily obtain the assertion (26) of Theorem 5. □

Corollary 4. *Let a function* f *belong to the class* $\mathcal{TW}^{\eta}_{0}\,(p,k;\phi,\varphi;A,B)$ *and let the sequence* $\{d_n\}$ *be defined by* (11). *If*

$$(28)\qquad d_k\le d_n\quad(n=k,k+1,\ldots),$$

then

$$(29)\qquad r^{-p}-\frac{B-A}{d_k}r^k\le|f(z)|\le r^{-p}+\frac{B-A}{d_k}r^k\quad(|z|=r<1).$$

Moreover, if

$$nd_k \le kd_n \quad (n = k, k+1, \ldots), \tag{30}$$

then

$$\begin{aligned} pr^{-p-1} - &\frac{k(B-A)}{d_k} r^{k-1} \\ &\le \left|f'(z)\right| \le pr^{-p-1} + \frac{k(B-A)}{d_k} r^{k-1} \quad (|z| = r < 1). \end{aligned} \tag{31}$$

The result is sharp, with the extremal function $f_{k,\eta}$ *of the form* (21).

4. The radius of starlikeness.

Theorem 6. *The radius of starlikeness of order* α *for the class* $\mathcal{TW}^{\eta}(p, k; \phi, \varphi; A, B)$ *is given by*

$$R_{\alpha}^{*}\left(\mathcal{TW}^{\eta}(p, k; \phi, \varphi; A, B)\right) = \inf_{n \ge k} \left(\frac{(p-\alpha)\, d_n}{(n-\alpha)(B-A)}\right)^{\frac{1}{n+p}}, \tag{32}$$

where d_n *is defined by* (11). *The functions* $f_{n,\eta}$ *of the form*

$$f_{n,\eta}(z) = a_{-p}\left(z^{-p} + \frac{B-A}{d_n} e^{-i(p+n)\eta} z^n\right) \tag{33}$$

$(z \in \mathcal{D}; n = k, k+1, \ldots; a_{-p} > 0)$ *are the extremal functions.*

Proof. A function $f \in \mathcal{T}^{\eta}(p, k)$ of the form (1) is starlike of order α in $\mathcal{D}(r)$, $0 < r \le 1$, if and only if it satisfies the condition (7). Since

$$\left|\frac{zf'(z)}{f(z)} + p\right| = \left|\frac{\sum_{n=k}^{\infty} (n+p) a_n z^n}{a_{-p} z^{-p} + \sum_{n=k}^{\infty} a_n z^n}\right| \le \frac{\sum_{n=k}^{\infty} (n+p)\, |a_n|\, |z|^{n+p}}{a_{-p} - \sum_{n=k}^{\infty} |a_n|\, |z|^{n+p}},$$

putting $|z| = r$, the condition (7) is true if

$$\sum_{n=k}^{\infty} \frac{n-\alpha}{p-\alpha} |a_n|\, r^{n+p} \le a_{-p}. \tag{34}$$

By Theorem 2, we have

$$\sum_{n=k}^{\infty} \frac{d_n}{B-A} |a_n| \le a_{-p}. \tag{35}$$

Thus, the condition (34) is true if

$$\frac{n-\alpha}{p-\alpha} r^{n+p} \le \frac{d_n}{B-A} \quad (n = k, k+1, \ldots),$$

that is, if

$$r \leq \left(\frac{(p-\alpha)\, d_n}{(n-\alpha)\,(B-A)} \right)^{\frac{1}{n+p}} \quad (n = k, k+1, \ldots). \tag{36}$$

It follows that each function $f \in \mathcal{TW}^{\eta}(p, k; \phi, \varphi; A, B)$ is starlike of order α in $\mathcal{D}(r)$, where

$$r = \inf_{n \geq k} \left(\frac{(p-\alpha)\, d_n}{(n-\alpha)\,(B-A)} \right)^{\frac{1}{n+p}}.$$

The functions $f_{n,\eta}$ of the form (33) realize the equality in (35), and the radius r cannot be larger. Thus we have (32). □

By Theorem 6 we have the following result.

Corollary 5. *Let the sequence $\{d_n + (B-A)\,\rho^{n+p}\}$, where $\{d_n\}$ is defined by* (11)*, be positive. The radius of starlikeness of order α for the class $\mathcal{TW}^{\eta}_{\rho}(p, k; \phi, \varphi; A, B)$ is given by*

$$R^{*}_{\alpha}\left(\mathcal{TW}^{\eta}_{\rho}(p, k; \phi, \varphi; A, B) \right) = \inf_{n \geq k} \left(\frac{(p-\alpha)\, d_n}{(n-\alpha)\,(B-A)} \right)^{\frac{1}{n+p}}.$$

The functions $f_{n,\eta}$ of the form (21) *are the extremal functions.*

5. Subordination results. Before stating and proving our subordination theorems for the class $\mathcal{TW}^{\eta}(p, k; \phi, \varphi; A, B)$ we need the following definition and lemma:

Definition 1. A sequence $\{b_n\}$ of complex numbers is said to be a subordinating factor sequence if for each function f of the form (1) from the class $\mathcal{S}^c$ we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \quad (a_1 = 1). \tag{37}$$

Lemma 3 ([14]). *A sequence $\{b_n\}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{ 1 + 2 \sum_{n=1}^{\infty} b_n z^n \right\} > 0 \quad (z \in \mathcal{U}). \tag{38}$$

Theorem 7. *Let the sequence $\{d_n\}$, defined by* (11)*, satisfy the inequality* (22)*. If $g \in \mathcal{S}^c$ and $f \in \mathcal{TW}^{\eta}(p, k; \phi, \varphi; A, B)$, then*

$$\left[\varepsilon z^{p+1} f(z) \right] * g(z) \prec g(z) \tag{39}$$

and

$$\operatorname{Re}\left[z^{p+1} f(z) \right] > -\frac{1}{2\varepsilon} \quad (z \in \mathcal{D}), \tag{40}$$

where

$$\varepsilon = \frac{d_k}{2a_{-p}\left(B - A + d_k\right)}. \tag{41}$$

If p and k are odd, and $\eta = 0$, then the constant factor ε cannot be replaced by a larger number.

Proof. Let a function f of the form (1) belong to the class $\mathcal{TW}^{\eta}\left(p, k; \phi, \varphi; A, B\right)$ and suppose that a function g of the form

$$g(z) = \sum_{n=1}^{\infty} c_n z^n \quad \left(c_1 = 1;\ z \in \mathcal{U}\right)$$

belongs to the class $\mathcal{S}^c$. Then

$$\left[\varepsilon z^{p+1} f\left(z\right)\right] * g(z) = \sum_{n=1}^{\infty} b_n c_n z^n \quad \left(z \in \mathcal{D}\right),$$

where

$$b_n = \begin{cases} \varepsilon a_{-p} & \text{if} \quad n = 1 \\ 0 & \text{if} \quad 2 \leq n \leq k + p \\ \varepsilon a_{n+p+1} & \text{if} \quad n > k + p. \end{cases}$$

Thus, by Definition 1 the subordination result (39) holds true if $\{b_n\}$ is the subordinating factor sequence. By Lemma 2 we have

$$\begin{aligned} \operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} &= \operatorname{Re}\left\{1 + 2\varepsilon a_{-p} z + \sum_{n=k}^{\infty} \frac{d_k}{B - A + d_k} a_n z^{n+p}\right\} \\ &\geq 1 - 2\varepsilon a_{-p} r - \frac{r}{\left(B - A + d_k\right) a_{-p}} \sum_{n=k}^{\infty} d_n \left|a_n\right| \quad \left(|z| = r < 1\right). \end{aligned}$$

Thus, by using Theorem 2 we obtain

$$\operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} \geq 1 - \frac{d_k}{B - A + d_k} r - \frac{B - A}{B - A + d_k} r > 0.$$

This evidently proves the inequality (38) and hence the subordination result (39). The inequality (40) follows from (39) by taking

$$g(z) = \frac{z}{1 - z} = \sum_{n=1}^{\infty} z^n \quad \left(z \in \mathcal{U}\right).$$

Next, we observe that the function $f_{k,\eta}$ of the form (33) belongs to the class $\mathcal{TW}^{\eta}\left(p, k; \phi, \varphi; A, B\right)$. If p and k are odd, and $\eta = 0$, then

$$z^{p+1} f_{k,0}\left(z\right)\Big|_{z=-1} = -\frac{1}{2\varepsilon},$$

and the constant (41) cannot be replaced by any larger one. □

Remark 2. By using (16) in Theorem 7 we obtain the result related to the class $\mathcal{TW}_{\rho}^{\eta}(p,k;\phi,\varphi;A,B)$. Moreover, by putting $\rho=0$, we have the following corollary.

Corollary 6. *Let the sequence* $\{d_n\}$ *defined by* (11) *satisfy the inequality* (22). *If* $g\in\mathcal{S}^c$ *and* $f\in\mathcal{TW}_{0}^{\eta}(p,k;\phi,\varphi;A,B)$, *then conditions* (39) *and* (40), *with*

$$\varepsilon=\frac{d_k}{2\,(B-A+d_k)} \tag{42}$$

hold true. If p *and* k *are odd, and* $\eta=0$, *then the constant factor* ε *in* (42) *cannot be replaced by a larger number.*

6. Partial sums. Let f be a function of the form (1). Due to Silverman [9] and Silvia [11] we investigate the partial sums f_m of the function f defined by

$$f_{k-1}(z)=a_{-p}z^{-p};\ \text{and}\ f_m(z)=a_{-p}z^{-p}+\sum_{n=k}^{m}a_nz^n,\ (m=k,k+1,\ldots). \tag{43}$$

In this section we consider partial sums of functions in the class $\mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B)$ and obtain sharp lower bounds for the ratios of real part of f to f_m and f' to f'_m.

Theorem 8. *Let the sequence* $\{d_n\}$ *defined by* (11) *be increasing and*

$$d_k\geq B-A. \tag{44}$$

If a function f *of the form* (1) *belongs to the class* $\mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B)$, *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_m(z)}\right\}\geq 1-\frac{B-A}{d_{m+1}}\qquad (z\in\mathcal{D},\ m=k-1,k,\ldots) \tag{45}$$

and

$$\operatorname{Re}\left\{\frac{f_m(z)}{f(z)}\right\}\geq \frac{d_{m+1}}{B-A+d_{m+1}}\qquad (z\in\mathcal{D},\ m=k-1,k,\ldots)\,. \tag{46}$$

The bounds are sharp, with the extremal functions $f_{n,\eta}$ *defined by* (33).

Proof. Since

$$\frac{d_{n+1}}{B-A}>\frac{d_n}{B-A}>1\qquad (n=k,k+1,\ldots)\,,$$

by Theorem 1 we have

$$\sum_{n=k}^{m}|a_n|+\frac{d_{m+1}}{B-A}\sum_{n=m+1}^{\infty}|a_n|\leq\sum_{n=k}^{\infty}\frac{d_n}{B-A}|a_n|\leq a_{-p}. \tag{47}$$

Let

$$(48)\quad g(z)=\frac{d_{m+1}}{B-A}\left\{\frac{f(z)}{f_m(z)}-\left(1-\frac{B-A}{d_{m+1}}\right)\right\}=1+\frac{\frac{d_{m+1}}{B-A}\sum\limits_{n=m+1}^{\infty}a_nz^{n+p}}{a_{-p}+\sum\limits_{n=k}^{m}a_nz^{n+p}}$$

$(z\in\mathcal{D})$. Applying (47), we find

$$\left|\frac{g(z)-1}{g(z)+1}\right|\leq\frac{\frac{d_{m+1}}{B-A}\sum\limits_{n=m+1}^{\infty}|a_n|}{2a_{-p}-2\sum\limits_{n=2}^{n}|a_n|-\frac{d_{m+1}}{B-A}\sum\limits_{n=m+1}^{\infty}|a_n|}\leq 1\quad(z\in\mathcal{D})\,.$$

Thus, we have

$$\operatorname{Re} g(z)\geq 0\quad(z\in U)\,,$$

which by (48) readily yields the assertion (45) of Theorem 8. Similarly, if we take

$$h(z)=\left(1+\frac{d_{m+1}}{B-A}\right)\left\{\frac{f_m(z)}{f(z)}-\frac{d_{m+1}}{B-A+d_{m+1}}\right\}\quad(z\in\mathcal{D})$$

and making use of (47), we can deduce that

$$\left|\frac{h(z)-1}{h(z)+1}\right|\leq\frac{\left(1+\frac{d_{m+1}}{B-A}\right)\sum\limits_{n=m+1}^{\infty}|a_n|}{2a_{-p}-2\sum\limits_{n=k}^{m}|a_n|-\left(\frac{d_{m+1}}{B-A}-1\right)\sum\limits_{n=m+1}^{\infty}|a_n|}\leq 1\quad(z\in\mathcal{D})\,,$$

which leads us immediately to the assertion (46) of Theorem 8. In order to see that the function $f_{m+1,\eta}$ of the form (21) gives the results sharp, we observe that

$$\frac{f_{m+1,\eta}(z)}{\left(f_{m+1,\eta}\right)_m(z)}=1-\frac{B-A}{d_{m+1}}\quad\left(z=e^{i\eta}\right),$$
$$\frac{\left(f_{m+1,\eta}\right)_m(z)}{f_{m+1,\eta}(z)}=\frac{d_{m+1}}{B-A+d_{m+1}}\quad\left(z=e^{i\left(\eta+\frac{\pi}{m+p+1}\right)}\right).$$

This completes the proof. □

Theorem 9. *Let the sequence* $\{d_n\}$ *defined by* (11) *be increasing and*

$$d_k>(m+1)\,(B-A)\,.$$

If a function f *of the form* (1) *belongs to the class* $\mathcal{TW}^{\eta}\,(p,k;\phi,\varphi;A,B)$, *then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f'_m(z)}\right\}\geq 1-\frac{(m+1)\,(B-A)}{d_{m+1}}\quad(z\in\mathcal{D},\ m=k-1,k,\dots)\,,$$
$$\operatorname{Re}\left\{\frac{f'_m(z)}{f'(z)}\right\}\geq\frac{d_{m+1}}{(m+1)\,(B-A)+d_{m+1}}\quad(z\in\mathcal{D},\ m=k-1,k,\dots)\,.$$

The bounds are sharp, with the extremal functions $f_{n,\eta}$ defined by (33).

Proof. By setting

$$g(z)=\frac{d_{m+1}}{B-A}\left\{\frac{f'(z)}{f_m'(z)}-\left(1-\frac{(m+1)(B-A)}{d_{m+1}}\right)\right\}\quad (z\in\mathcal{D}),$$

and

$$h(z)=\left(m+1+\frac{d_{m+1}}{B-A}\right)\left\{\frac{f_m'(z)}{f'(z)}-\frac{d_{m+1}}{(m+1)(B-A)+d_{m+1}}\right\}\quad (z\in\mathcal{D}),$$

the proof is analogous to that of Theorem 8, and we omit the details. □

Remark 3. By using (16) in Theorems 8 and 9 we obtain the results related to the class $\mathcal{TW}_{\rho}^{\eta}(p,k;\phi,\varphi;A,B)$.

7. Concluding remarks. We conclude this paper by observing that, in view of the subordination relation (9), choosing the functions ϕ and φ, we can consider new and also well-known classes of functions. In particular, the class

$$\mathcal{W}^{n}(p,k;\varphi;A,B):=\mathcal{W}\left(p,k;\frac{z\varphi'(z)}{-p},\sum_{l=0}^{n-1}\varphi\left(x^{l}z\right);A,B\right),$$

where $n\in\mathbb{N}$, $x^n=1$ contains functions $f\in\mathcal{A}(p,k)$, which satisfy the condition

$$\frac{z(\varphi*f)'(z)}{\sum_{l=0}^{n-1}(\varphi*f)(x^{l}z)}\prec -p\frac{1+Az}{1+Bz}.$$

It is related to the class of starlike functions with respect to n-symmetric points. Moreover putting $n=1$, we obtain the class $\mathcal{W}^{1}(p,k;\varphi;A,B)$ defined by the following condition

$$\frac{z(\varphi*f)'(z)}{(\varphi*f)(z)}\prec -p\frac{1+Az}{1+Bz}.$$

In particular, a function $f\in\mathcal{M}$ belongs to the class

$$\mathcal{W}(p,k;\varphi;\alpha):=\mathcal{W}^{1}(p,k;\varphi;2\alpha-1,1)\quad (0\le\alpha<p)$$

if it satisfies the condition

$$\operatorname{Re}\left\{-\frac{z(\varphi*f)'(z)}{(\varphi*f)(z)}\right\}>\alpha\quad (z\in\mathcal{D}).$$

The class $\mathcal{W}(p,k;\varphi;\alpha)$ is related to the class of meromorphic multivalent starlike function of order α. In particular, we have

$$\mathcal{MS}_{p}^{*}(\alpha):=\mathcal{W}\left(p,p+1;\frac{1}{z^{p}(1-z)};\alpha\right).$$

Let λ be a complex parameter. A function $f \in \mathcal{A}(p,k)$ belongs to the class

$$\mathcal{V}_\lambda(p,k;\varphi;A,B) \coloneqq \mathcal{W}\left(p,k;\lambda\varphi(z)+(1-\lambda)\frac{z\varphi'(z)}{-p},z^{-p};A,B\right)$$

if it satisfies the following condition:

$$\lambda z^p(\varphi * f)(z)+(1-\lambda)\frac{z^{p+1}(\varphi * f)'(z)}{-p} \prec \frac{1+Az}{1+Bz}.$$

The considered classes are defined by using the convolution $\varphi * f$ or equivalently by the linear operator

$$J_\varphi : \mathcal{M}(p,k) \to \mathcal{M}(p,k), \quad J_\varphi(f) = \varphi * f.$$

By choosing the function φ, we can obtain a lot of important linear operators, and in consequence new and also well-known classes of functions. We can list here some of these linear operators related to the Sălăgean operator, the Cho–Kim–Srivastava operator, the Dziok–Raina operator, the Hohlov operator, the Dziok–Srivastava operator, the Carlson–Shaffer operator, the Ruscheweyh derivative operator, the generalized Bernardi–Libera–Livingston operator, the fractional derivative operator, and so on (see, for the precise relationships [6]).

If we apply the results presented in this paper to the classes discussed above, we can obtain several additional results. Some of these results were obtained in earlier works, see for example [1, 2, 4, 5, 8, 12, 13].

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