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Extended fractional calculus of variations, complexified geodesics and Wong's fractional equations on complex plane and on Lie algebroids

Abstract. In this work, we communicate the topic of complex Lie algebroids based on the extended fractional calculus of variations in the complex plane. The complexified Euler–Lagrange geodesics and Wong's fractional equations are derived. Many interesting consequences are explored.

1. Introduction. The concept of fractional derivative and integral dates from the origin of calculus itself. It was founded in fact by Liouville between 1832 and 1837. However, in 1876, Riemann introduced the fractional integral and accordingly, the definition of Riemann–Liouville (RL) fractional integro-differentiation was recognized and has become most widely used. Nevertheless, when dealing with a dynamical system, we are unable to apply non-zero initial conditions with real life measurable quantities making use of the RL fractional integral. This was done only in 1967 by Caputo, who he introduced the well-known Caputo fractional derivative in a way similar to RL, but only with a change of the order of integration and differentiation. Caputo fractional derivative has become lately well-liked in the area of algorithm development. The reader is referred to Podlubny book for a summary of basic techniques, geometrical and physical interpretation

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of fractional derivatives and integrals [57]. It is noteworthy that fractional calculus plays a crucial role in dynamical systems governed by fractional differential equations as the memory effect of the convolution of the fractional integral gives the equation amplified significant power [54, 59, 53]. This amazing fact was proved to play a crucial role in different branches of sciences and engineering, including viscoelasticity, viscoplasticity, damping and diffusion [42, 60, 55, 3, 56, 61, 62]. Furthermore, fractional derivatives have memory effect and hence one naturally expects interesting consequences in modeling behaviour of complex classical and quantum dynamical systems [36, 35, 37, 40, 41].

On the other hand, in recent years, plentiful works have been dedicated to one important branch of fractional calculus which is the "Fractional Calculus of Variations" (FCV). We believe today after a series of papers that the FCV play a leading role in nonconservative dynamical system where traditional Lagrangian and Hamiltonian mechanics based on the standard calculus of variations do not hold and fail to describe correctly the dissipative behavior [58, 1, 2, 5, 6, 25, 24]. Nevertheless, the largest part of the approaches discussed in literature deal with Riemann–Liouville derivatives and integrals, e.g. the Fractional ActionLike Variational Approach (FALVA) [9, 10] and a few with Caputo or Riesz derivatives and integrals [31]. Depending on the category of functional integral considered, different fractional Euler-Lagrange type equations were obtained. Necessary optimality conditions of Euler-Lagrange type for the fundamental fractional problem of the calculus of variations were obtained and discussed under appropriate convexity assumptions [28, 27, 29, 30, 32, 4]. However, we believe that the concept of FCV could be introduced in a number of ways making use of different types of fractional operators, e.g. Erdélyi–Kober integral, Saxena–Kumbhat hypergeometric integral, generalized single-time Stieltjes fractal-fractional integral, extended exponentially fractional integral and so on [12, 23, 20, 21]. These new concepts were proved to play a critical role in open one-dimensional dynamical systems with only one degree of freedom exhibiting an exponentially increasing or decaying mass and varying periodical frequency, e.g. quantum Bateman–Feshbach–Tikochinsky and Caldirola–Kanai damped harmonic oscillators [16, 17].

Further extended scenario of FCV was introduced more recently by the author of the present paper and entitled "periodic functional approach" based on a periodic fractional kernel inside the integral action [21]. The new approach was proved to play a vital role in modern cosmology. In this work, we introduce a new kind of extended exponentially FCV with complex fractional derivatives. Our main aim is to derive the fractional Euler–Lagrange equation on complex Lie algebroids, i.e. complex plane. The notion of Lie algebroid is in reality a generalization of both the concept of Lie algebra and the concept of an integrable distribution. The category of Lie algebroids has proved to be useful in the formulation and analysis of numerous problems in differential geometry.

Given that a Lie algebroid is a notion which merely unifies tangent bundles and Lie algebras, one expects their relation to Lagrangian and Hamiltonian classical mechanics. In the context of classical mechanics, Weinstein was the first who developed a theory of Lagrangian and Hamiltonian systems on Lie algebroids and their discrete analog on Lie groupoids [63, 64]. In the last years, Weinstein's theory has been developed by many authors, and accordingly, an influential mathematical structure was emerged and ranged from classical to generalized complex Lie algebroids with vakonomic and nonholonomic constraints. Weinstein's theory plays as well a crucial role in symmetry problems [50, 51, 52]. Symmetries are mathematically described by parameter groups of transformations. Their significance ranges from elementary and theoretical aspects to physical applications, having thoughtful implications in the dynamical behavior of the systems and in their basic qualitative properties. Nevertheless, we are also aware of the importance of symmetry issue in Lie algebroids throughout mathematics and of the fact that the manifolds of Lie groups and their homogeneous spaces provide accessible examples on which to do geometry and analysis. This is made promising by the fact that Lie groups may be studied in terms of their Lie algebras: the Lie algebra is a linear space and the entire linear algebra techniques may be brought to stand on it. This issue plays also a crucial role in Connes's noncommutative geometry [7, 8, 46]. It is noteworthy that Lie algebroids generalize Lie algebras, and in certain quantum systems they represent extended quantum (algebroid) symmetries.

However, a basic motivation for the study of fractional calculus of variations on complex Lie algebroids is a description of the renowned results for some moduli spaces of holomorphic structures on a complex vector bundle over a compact complex manifold in a unified handling, e.g. the related Hitchin–Kobayashi correspondence and the moduli space of Higgs bundles on a compact Riemann surface [49, 33, 34, 38, 50, 44]. More recent examples of this phenomenon involve the moduli space of complex B-branes and the moduli space of symplectic A-branes, based on generalized complex geometry. They play an extremely important role in mirror symmetry and geometric Langland's program [47, 48]. These reasons motivate to find an extended fractional action-like variational description of Lagrange's equations for a Lagrangian system defined on a more general Lie algebroid.

The paper is organized as follows: in Section 2, we introduce the basic concepts of extended exponentially FCV with complex fractional derivative on complex plane. In Section 3, we introduce the Lagrangian concepts on complex Lie algebroids. The resulting complexified Euler–Lagrange, geodesics and Wong's equations are discussed in the same section. In Section 4, we initiate the complexified Hamiltonian formalism on complex Lie algebroids. Conclusions and perspectives are discussed in Section 5.

2. Extended complexified fractional action-like variational approach. We start by introducing the following definition:

Definition 2.1 (Extended Complexified Fractional Integral (ECFI)). Let f be an analytic function in a simply connected region of the z-plane containing the origin. The ECFI of order $\alpha > 0$, is defined by:

$$K_{(z)}^{(-\alpha)} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(\cosh z - \cosh\zeta)^{\alpha-1} d\zeta, \tag{1}$$

where the multiplicity of $(\cosh z - \cosh\zeta)^{\alpha-1}$ is removed by requiring $\log(\cosh z - \cosh\zeta)$ to be real when $\cosh z - \cosh\zeta > 0$. Here $\Gamma(\alpha)$ is the Euler gamma function [26].

Definition 2.2. Let f be an analytic function in a simply connected region of the z-plane containing the origin. The Riesz–Caputo complex fractional derivative of order $0 \le \alpha < 1$, is defined by:

$$D_{(z)}^{(\alpha)} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z (\cosh z - \cosh\zeta)^{-\alpha} f(\zeta) d\zeta, \tag{2}$$

where the multiplicity of $(\cosh z - \cosh\zeta)^{\alpha-1}$ is removed by requiring $\log(\cosh z - \cosh\zeta)$ to be real when $\cosh z - \cosh\zeta > 0$.

Definition 2.3. Given a smooth Lagrangian function in a simply connected region of the z-plane containing the origin assumed to be a C^2-function with respect to all its arguments. The extended complexified fractional action is defined by

$$S_{(z)}^{(-\alpha)} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z L(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta))(\cosh z - \cosh\zeta)^{\alpha-1} d\zeta \tag{3}$$

under given boundary condition $q(0) = q_0$.

To define the corresponding extended fractional variational action and consequently the fractional variational problem on real or complex Lie algebroids, we have to identify two basic objects: an infinite-dimensional manifold M of paths whose tangent space TM represents all possible variations and an action functional on M [50, 51, 52]. Then, we have to choose a submanifold of admissible paths and a set of admissible variations of admissible paths.

In fact, given a vector bundle $\pi : E \to M$ given by a vector bundle map $\rho : E \to TM$ over the identity in M called the anchor map, a smooth Lagrangian function $L : E \to \mathbb{R}$, together with a Lie algebra structure on the $C^\infty(M)$-module of sections of E determined by the Lie bracket. In turn, the Lie bracket induces a Lie algebra homomorphism $\bar{\rho}$ of $C^{\infty}(M)$ from $\mathrm{Sec}(E) \to \chi(M)$ by the anchor map $\rho : E \to TM$ given by:

$$\sigma \in \mathrm{Sec}(E) \to \bar{\rho}(\sigma)(q) \in \chi(M)$$
$$\bar{\rho}(\sigma)(q) = \rho(s(q)),\ \forall q \in M$$

satisfying the Leibniz compatibility identity:

$$[\sigma, f\eta]_E = (\rho(\sigma)f)\eta + f[\sigma,\eta]_E,\ \forall\ f \in C^{\infty}(M),\ \sigma,\eta \in \mathrm{Sec}(E).$$

Definition 2.4. A vector bundle (E, ξ, M) endowed with a Lie algebroid structure $([\cdot,\cdot]_E, \rho)$ is called Lie algebroid over M and is denoted by the triple $(E, [\cdot,\cdot]_E, \rho)$.

A local coordinate system (q^i) in the base manifold M, and a local basis of section (e_γ) of E determine a local coordinate system (q^i, y^η) on E. The anchor and the bracket are locally determined by the structure functions

$$\rho_i^\gamma, C_{ij}^k \in C^{\infty}(M) \text{ of } (E, [\cdot,\cdot]_E, \rho)$$

with

$$\bar{\rho}(e_i) = \rho_i^\eta \frac{\partial}{\partial q^n} \equiv \rho_i^\eta \partial_\eta,$$
$$\partial_\eta \doteq \frac{\partial}{\partial q^n},$$
$$[e_i, e_j] = C_{ij}^k e_k,$$

which satisfies the following structure relations, which results from the Leibniz identity and the Jacobi identity:

$$\rho_i^\eta \frac{\partial \rho_j^\sigma}{\partial q^\eta} - \rho_j^\eta \frac{\partial \rho_i^\sigma}{\partial q^\eta} = \rho_k^\sigma C_{ij}^k,\ \eta = \overline{1,N},\ i = \overline{1,N}, \tag{4}$$

$$\sum_{\text{cyclic } (i,j,k)} \left(\rho_i^\eta \frac{\partial C_{jk}^l}{\partial q^\eta} + C_{jk}^h C_{ih}^l \right) = 0. \tag{5}$$

This innovative formalism was proved in recent years to be a powerful tool to investigate many fundamental properties of a given complex dynamical systems. Complex Lie algebroid structures on complex vector bundles over real bases can be defined similarly, replacing the tangent bundle of the base by the complexified tangent bundle. We consider at the moment the space of E-paths on the complex Lie algebroid denoted by $\mathcal{P}(J, E),\ J = [0, z]$, which is a differentiable Banach manifold.

Theorem 2.1 ([43]). *Let the manifold M be the space of all C^1-paths $[0, z] \to E$ in E and $L : E \to \mathbb{C}$ be a smooth Lagrangian function in a simply connected region of the z-plane containing the origin assumed to be a C^2-function with respect to all its arguments on the complex Lie algebroid E with admissible curve q in E and with two fixed endpoints*

$$A, B \in M \subset \mathcal{P}(J, E)_A^B = \{\mathcal{P}(J, E) | \pi(q(0)) = A \text{ and } \pi(q(z)) = B\}.$$

Let $\tau_L = (\partial L/\partial y^i)dy^i$ *in local coordinate. The critical points of the complexified fractional action integral* $I : \mathcal{P}(J, E) \to \mathbb{R}$ *defined by the extended fractional complexified action* (3) *on the complex Banach manifold* $\mathcal{P}(J, E)_A^B$ *are exactly those elements of that space which satisfy the following extended complexified fractional Euler–Lagrange equation:*

$$\delta^r L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right) = 0,$$

where

$$\begin{aligned}\left\langle \delta^r L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right)\right\rangle &= \left\langle dL\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right), \sum_q (\sigma)\right\rangle \\ &\quad - D_{(z)}^{(\alpha)} \left\langle z_{L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right)} \circ q, \sigma\right\rangle \\ &\quad - \left[\frac{(\alpha-1)\sinh\zeta}{\cosh z - \cosh\zeta}\right]\left\langle z_{L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right)} \circ q, \sigma\right\rangle.\end{aligned} \tag{6}$$

Proof. Let $q \in \mathcal{P}(J, E)_A^B$, the tangent space on $\mathcal{P}(J, E)_A^B$ is the set of complex vector fields along q of the form:

$$\Xi_q(\sigma) : \sigma \in \mathrm{Sec}(E)/\sigma(a) = \sigma(b) = 0.$$

The action (3) is smooth, then

$$\begin{aligned}0 &= \left\langle dS_{(z)}^{(-\alpha)}, \Xi_q(f\sigma)\right\rangle \\ &= \frac{1}{\Gamma(\alpha)} \int_0^z \left\langle L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right), \Xi_q(f\sigma)\right\rangle (\cosh z - \cosh\zeta)^{\alpha-1} d\zeta \\ &= \frac{1}{\Gamma(\alpha)} \left(\int_0^z \left[f(\zeta)\left\langle dL\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right), \Xi_q(\sigma)\right\rangle\right.\right. \\ &\quad - D_{(z)}^{(\alpha)} \frac{d}{d\zeta} \left\langle z_{L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right)} \circ q, \sigma\right\rangle (\cosh z - \cosh\zeta)^{\alpha-1} \\ &\quad \left. + \frac{d}{d\zeta}\left[(\cosh z - \cosh\zeta)^{\alpha-1}\right]\left\langle \tau_{L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right)} \circ q, \sigma\right\rangle\right] d\zeta \\ &\quad \left. + f(\zeta)\left\langle \tau_{L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right)} \circ q, \sigma\right\rangle (\cosh z - \cosh\zeta)^{\alpha-1}\Big|_0^z\right).\end{aligned}$$

Using the fact that

$$\Xi_q(f(\zeta)\sigma(\zeta)) = f(\zeta)\sum_q(\sigma(\zeta)) + \frac{df(\zeta)}{d\zeta}\sigma_q^v$$

we obtain after simple algebraic manipulation

$$\left\langle dS_{(z)}^{(-\alpha)}, \Xi_q(f(\zeta)\sigma(\zeta))\right\rangle = \int_0^z f(\zeta)\left\langle \delta^r L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right), \sigma(\zeta)\right\rangle d\zeta.$$

Here $v = \rho(q)$ is the actual velocity and $d\sigma_q^v/d\zeta$ is the Riemann–Liouville complex fractional derivative of the canonical vertical lift of σ. This equation is satisfied for every analytic function defined in a simply connected region of the z-plane containing the origin and for every section $\sigma \in \mathrm{Sec}(E)$. Thus, the critical points are satisfied by

$$\delta^r L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right) = 0. \qquad \square$$

Corollary 2.1. *The fractional complex Euler–Lagrange equation in complex local coordinates is*

$$\begin{aligned} &\rho_i^\eta \frac{\partial L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right)}{\partial x^\eta} - D_{(z)}^{(\alpha)} \frac{\partial L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right)}{\partial y^i} \\ &\qquad - \frac{\partial L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right)}{\partial y^k} C_{ij}^k y^j \\ &\qquad - \frac{(\alpha-1)\sinh\zeta}{\cosh z - \cosh\zeta} \frac{\partial L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right)}{\partial y^i} = 0, \end{aligned} \tag{7}$$

where

$$\dot{x}^\eta \equiv \frac{dx^\eta}{d\zeta} = \rho_i^\eta y^i. \tag{8}$$

Remark 2.1. If for instance we set $\zeta^\eta = \zeta_1^\eta + i\zeta_2^\eta$, $x^\eta = x_1^\eta + ix_2^\eta$ and $y^j = y_1^j + iy_2^j$, then the complexified Lagrangian is split into a real and a complex part as follows: $L = L_1 + iL_2$. The following rules are useful:

$$\frac{\partial}{\partial x^\eta} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^\eta} - i\frac{\partial}{\partial x_2^\eta}\right), \tag{9}$$

$$\frac{\partial}{\partial y^i} = \frac{1}{2}\left(\frac{\partial}{\partial y_1^i} - i\frac{\partial}{\partial y_2^i}\right), \tag{10}$$

$$\begin{aligned} \rho_i^\eta \frac{\partial \rho_j^\sigma}{\partial q_1^\eta} - \rho_j^\eta \frac{\partial \rho_i^\sigma}{\partial q_1^\eta} - i\left(\rho_i^\eta \frac{\partial \rho_j^\sigma}{\partial q_2^\eta} - \rho_j^\eta \frac{\partial \rho_i^\sigma}{\partial q_2^\eta}\right) &\equiv \left[\rho_k^\sigma C_{ij}^k\right]_1 - i\left[\rho_k^\sigma C_{ij}^k\right]_2 \\ &\equiv \rho_k^\sigma\left(\Re\left[C_{ij}^k\right]_1 - i\Im\left[C_{ij}^k\right]_2\right) \equiv \rho_k^\sigma \tilde{C}_{ij}^k, \end{aligned} \tag{11}$$

where

$$\tilde{C}_{ij}^k = \Re\left[C_{ij}^k\right]_1 - i\Im\left[C_{ij}^k\right]_2 \equiv \overline{C}_{ij}^k - i\underline{C}_{ij}^k. \tag{12}$$

Remark 2.2. By letting $z = x + iy$ and $\zeta = \zeta_1 + i\zeta_2$, we may split the Riesz–Caputo complex fractional derivative into a real and a complex part as follows:

$$\begin{aligned}D_{(z)}^{(\alpha)}f(x) &= \frac{1}{2\Gamma(1-\alpha)}\left(\frac{d}{dx}-i\frac{d}{dy}\right)\\ &\quad\times\int_0^{x+iy}(\cosh(x+iy)-\cosh(\zeta_1+i\zeta_2))^{-\alpha}(F(\zeta_1)+iG(\zeta_2))(d\zeta_1+id\zeta_2)\\ &= \frac{1}{2\Gamma(1-\alpha)}\left(\frac{d}{dx}-i\frac{d}{dy}\right)\int_0^{x+iy}(\cosh(x+iy)-\cosh(\zeta_1+i\zeta_2))^{-\alpha}\\ &\quad\times\left[(F(\zeta_1)d\zeta_1-G(\zeta_2)d\zeta_2)+i(F(\zeta_1)d\zeta_2+G(\zeta_2)d\zeta_1)\right].\end{aligned}$$

Making use of the simple decomposition

$$\begin{aligned}&\cosh(x+iy)-\cosh(\zeta_1+i\zeta_2)\\ &\qquad=\cos y\cosh x-\cos\zeta_2\cosh\zeta_1+i(\sin y\sinh x-\sin\zeta_2\sinh\zeta_1),\end{aligned}$$

and the rule [65]

$$(X+iY)^{-\alpha}=R^{-\alpha}\exp(-iY\varphi\alpha)=R^{-\alpha}[\cos(\alpha Y\varphi)-i\sin(\alpha Y\varphi)],$$

where

$$R=\sqrt{X^2+Y^2},\quad \varphi=\frac{\pi}{2}-\arcsin\frac{X}{R},$$
$$X=\cosh y\cosh x-\cos\zeta_2\cosh\zeta_1,\quad Y=\sin y\sinh x-\sin\zeta_2\sinh\zeta_1,$$

we obtain after simple algebraic manipulation

$$\begin{aligned}D_{(z)}^{(\alpha)}f(x) &= \frac{R^{-\alpha}}{2\Gamma(1-\alpha)}\left[\frac{d}{dx}\int_0^{x+iy}[\cos(\alpha Y\varphi)(F(\zeta_1)d\zeta_1-G(\zeta_2)d\zeta_2)\right.\\ &\qquad+\sin(\alpha Y\varphi)(F(\zeta_1)d\zeta_2-G(\zeta_2)d\zeta_1)]\\ &\qquad+\frac{d}{dy}\int_0^{x+iy}[\cos(\alpha Y\varphi)(F(\zeta_1)d\zeta_2+G(\zeta_2)d\zeta_1)\\ &\qquad\left.-\sin(\alpha Y\varphi)(F(\zeta_1)d\zeta_1-G(\zeta_2)d\zeta_2)]\right]\\ &\quad+i\frac{R^{-\alpha}}{2\Gamma(1-\alpha)}\left[\frac{d}{dx}\int_0^{x+iy}[\cos(\alpha Y\varphi)(F(\zeta_1)d\zeta_2+G(\zeta_2)d\zeta_1)\right.\\ &\qquad-\sin(\alpha Y\varphi)(F(\zeta_1)d\zeta_1-G(\zeta_2)d\zeta_2)]\\ &\qquad-\frac{d}{dy}\int_0^{x+iy}[\cos(\alpha Y\varphi)(F(\zeta_1)d\zeta_1-G(\zeta_2)d\zeta_2)\\ &\qquad\left.+\sin(\alpha Y\varphi)(F(\zeta_1)d\zeta_2+G(\zeta_2)d\zeta_1)]\right].\end{aligned}\tag{13}$$

However,

$$\frac{d}{dy}\int_0^{x+iy}[\cos(\alpha Y\varphi)(F(\zeta_1)d\zeta_2+G(\zeta_2)d\zeta_1)-\sin(\alpha Y\varphi)(F(\zeta_1)d\zeta_1-G(\zeta_2)d\zeta_2)],$$

and

$$\frac{d}{dy}\int_0^{x+iy}[\cos(\alpha Y\varphi)(F(\zeta_1)d\zeta_1-G(\zeta_2)d\zeta_2)+\sin(\alpha Y\varphi)(F(\zeta_1)d\zeta_2+G(\zeta_2)d\zeta_1)],$$

will both acquire complex values and accordingly by denoting:

$$\begin{aligned}
&\frac{d}{dx}\int_0^{x+iy}[\cos(\alpha Y\varphi)(F(\zeta_1)d\zeta_1\\
&\qquad - G(\zeta_2)d\zeta_2)+\sin(\alpha Y\varphi)(F(\zeta_1)d\zeta_2+G(\zeta_2)d\zeta_1)] = D^{\alpha}_{1(x)}f,\\
&\frac{d}{dx}\int_0^{x+iy}[\cos(\alpha Y\varphi)(F(\zeta_1)d\zeta_2+G(\zeta_2)d\zeta_1)\\
&\qquad - \sin(\alpha Y\varphi)(F(\zeta_1)d\zeta_1-G(\zeta_2)d\zeta_2)] = D^{\alpha}_{2(x)}f,\\
&\frac{d}{dy}\int_0^{x+iy}[\cos(\alpha Y\varphi)(F(\zeta_1)d\zeta_2+G(\zeta_2)d\zeta_1)\\
&\qquad - \sin(\alpha Y\varphi)(F(\zeta_1)d\zeta_1-G(\zeta_2)d\zeta_2)] = iD^{\alpha}_{1(y)}f,\\
&\frac{d}{dy}\int_0^{x+iy}[\cos(\alpha Y\varphi)(F(\zeta_1)d\zeta_1-G(\zeta_2)d\zeta_2)\\
&\qquad + \sin(\alpha Y\varphi)(F(\zeta_1)d\zeta_2+G(\zeta_2)d\zeta_1)] = iD^{\alpha}_{2(y)}f,
\end{aligned}$$

equation (13) is reduced to

$$D^{(\alpha)}_{(z)}f(z)=\frac{R^{-\alpha}}{2\Gamma(1-\alpha)}\left[D^{(\alpha)}_{1(x)}f+D^{(\alpha)}_{2(y)}f\right]+i\frac{R^{-\alpha}}{2\Gamma(1-\alpha)}\left[D^{(\alpha)}_{1(y)}f+D^{(\alpha)}_{2(x)}f\right],$$

$$D^{(\alpha)}_{(z)}f(z)=\Re D^{(\alpha)}_{(z)}f(z)+i\Im D^{(\alpha)}_{(z)}f(z)=\overline{D}^{(\alpha)}_{(z)}f(z)+i\underline{D}^{(\alpha)}_{(z)}f(z), \tag{14}$$

where

$$\begin{aligned}
\Re D^{(\alpha)}_{(z)}f(z)\equiv\overline{D}^{(\alpha)}_{(z)}f(z)&=\frac{R^{-\alpha}}{2\Gamma(1-\alpha)}\left[D^{(\alpha)}_{1(x)}f+D^{(\alpha)}_{2(y)}f\right]\\
\Im D^{(\alpha)}_{(z)}f(z)\equiv\underline{D}^{(\alpha)}_{(z)}f(z)&=\frac{R^{-\alpha}}{2\Gamma(1-\alpha)}\left[D^{(\alpha)}_{1(y)}f+D^{(\alpha)}_{2(x)}f\right].
\end{aligned}$$

Lemma 2.1. *The fractional complexified Euler–Lagrange equation* (7) *is split into two parts:*

$$
\begin{aligned}
&\rho_i^\eta\left(\frac{\partial L_1}{\partial x_1^\eta}+\frac{\partial L_2}{x_2^\eta}\right)\\
&-\left(\overline{D}_{(z)}^{(\alpha)}f(z)\left(\frac{\partial L_1}{\partial y_1^j}+\frac{\partial L_2}{\partial y_2^j}\right)-\underline{D}_{(z)}^{(\alpha)}f(z)\left(\frac{\partial L_1}{\partial y_1^j}+\frac{\partial L_2}{\partial y_2^j}\right)\right)\\
&-\left(\frac{\partial L_1}{\partial y_1^k}+\frac{\partial L_2}{y_2^k}\right)\left(\overline{C}_{ij}^k y_1^j+\underline{C}_{ij}^k y_2^j\right)\\
&+\left(\frac{\partial L_2}{\partial y_1^k}-\frac{\partial L_1}{y_2^k}\right)\left(\overline{C}_{ij}^k y_2^j-\underline{C}_{ij}^k y_1^j\right)\\
&-(\alpha-1)\left[p\left(\frac{\partial L_1}{\partial y_1^k}+\frac{\partial L_2}{y_2^k}\right)-q\left(\frac{\partial L_2}{\partial y_1^k}-\frac{\partial L_1}{y_2^k}\right)\right]=0,
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
&\rho_i^\eta\left(\frac{\partial L_2}{\partial x_1^\eta}-\frac{\partial L_1}{x_2^\eta}\right)\\
&-\left(\overline{D}_{(z)}^{(\alpha)}f(z)\left(\frac{\partial L_2}{\partial y_1^j}-\frac{\partial L_1}{\partial y_2^j}\right)+\underline{D}_{(z)}^{(\alpha)}f(z)\left(\frac{\partial L_1}{\partial y_1^j}+\frac{\partial L_2}{\partial y_2^j}\right)\right)\\
&-\left(\frac{\partial L_2}{\partial y_1^k}-\frac{\partial L_1}{y_2^k}\right)\left(\overline{C}_{ij}^k y_1^j+\underline{C}_{ij}^k y_2^j\right)\\
&-\left(\frac{\partial L_1}{\partial y_1^k}+\frac{\partial L_2}{y_2^k}\right)\left(\overline{C}_{ij}^k y_2^j-\underline{C}_{ij}^k y_1^j\right)\\
&-(\alpha-1)\left[q\left(\frac{\partial L_1}{\partial y_1^k}+\frac{\partial L_2}{y_2^k}\right)+p\left(\frac{\partial L_2}{\partial y_1^k}-\frac{\partial L_1}{y_2^k}\right)\right]=0,
\end{aligned}
\tag{16}
$$

where

$$\sinh(\zeta_1+i\zeta_2)=\sinh\zeta_1\cos\zeta_2+i\cosh\zeta_1\sin\zeta_2=A+iB, \tag{17}$$

$$p=\frac{AX+BY}{X^2+Y^2}, \tag{18}$$

$$q=\frac{BX-AY}{X^2+Y^2}. \tag{19}$$

Remark 2.3. Making use of the previous arguments, one may already notice that the complexified action (3) may also be split into a real and a complexified part. We expect that the emergent imaginary part of the action will contribute as a corrector to the real part when applied to a physical problem and exotic solutions that will arise from the complexified counterpart may have physical relevance [18, 19].

Remark 2.4. We expect that equations (18) and (19) will play a crucial role when dealing with dissipative dynamical systems on complex Lie algebroids, in particular those dynamical systems exhibiting oscillatory motion.

Remark 2.5. For mathematical convenience and clarification, we will use in the rest of the manuscript the following notations:

$$\frac{\overline{\partial}}{\overline{\partial} x_1^\eta} \equiv \overline{D}^{(\alpha)},$$
$$\frac{\underline{\partial}}{\underline{\partial} x_1^\eta} \equiv \underline{D}^{(\alpha)},$$

where $\overline{D}^{(\alpha)}$ and $\underline{D}^{(\alpha)}$ are defined by equation (14).

As an exemplification, we will derive the fractional geodesics for complex Lie algebroids. In complexified coordinates, the complexified Lagrangian

$$L = \frac{1}{2} g_{ij}(x) y^i y^j = L_1 + iL_2,$$

is expected to induce an isomorphism of the complex vector (dual E^*) bundles $\tilde{g} : E \to E^*$. Here the metric

$$g = g_{ij}(x) e^i \otimes e^j,$$

is complexified, i.e. $g = g_1 + ig_2$. The fractional complexified Euler–Lagrange equations (15) and (16) now read:

$$\begin{aligned}
&\frac{1}{2}\frac{\overline{\partial}}{\overline{\partial} x_1^\eta}\left[g_{ik1}y_1^i - g_{ik2}y_2^i\right] + \frac{1}{2}\frac{\underline{\partial}}{\underline{\partial} x_2^\eta}\left[g_{ik1}y_1^i - g_{ik2}y_2^i\right] \\
&\quad = (\alpha-1)\left[p(g_{ik1}y_1^i - g_{ik2}y_2^i - q(g_{ik1}y_2^i + g_{ik2}y_1^i)\right] \\
&\qquad + \left[\overline{C}_{ik}^j g_{sj1} + \underline{C}_{ik}^j g_{sj2} + \frac{1}{4}\rho_k^\eta\left(\frac{\overline{\partial} g_{ij1}}{\overline{\partial} x_1^\eta} + \frac{\underline{\partial} g_{ij2}}{\underline{\partial} x_2^\eta}\right)\right]\left[y_1^i y_1^j - y_2^i y_2^j\right] \\
&\qquad - \left[\overline{C}_{ik}^j g_{sj2} + \underline{C}_{ik}^j g_{sj1} + \frac{1}{4}\rho_k^\eta\left(\frac{\overline{\partial} g_{ij2}}{\overline{\partial} x_1^\eta} - \frac{\underline{\partial} g_{ij1}}{\underline{\partial} x_2^\eta}\right)\right]\left[y_1^i y_2^j + y_2^i y_1^j\right]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{\overline{\partial}}{\overline{\partial} x_1^\eta}\left[g_{ik1}y_2^i + g_{ik2}y_1^i\right] - \frac{1}{2}\frac{\underline{\partial}}{\underline{\partial} x_2^\eta}\left[g_{ik1}y_1^i - g_{ik2}y_2^i\right] \\
&\quad = (\alpha-1)\left[p(g_{ik1}y_2^i + g_{ik2}y_1^i) + q(g_{ik1}y_1^i + g_{ik2}y_2^i)\right] \\
&\qquad + \left[\overline{C}_{ik}^j g_{sj2} - \underline{C}_{ik}^j g_{sj1} + \frac{1}{4}\rho_k^\eta\left(\frac{\overline{\partial} g_{ij2}}{\overline{\partial} x_1^\eta} - \frac{\underline{\partial} g_{ij1}}{\underline{\partial} x_2^\eta}\right)\right]\left[y_1^i y_1^j - y_2^i y_2^j\right] \\
&\qquad - \left[\overline{C}_{ik}^j g_{sj1} + \underline{C}_{ik}^j g_{sj2} + \frac{1}{4}\rho_k^\eta\left(\frac{\overline{\partial} g_{ij1}}{\overline{\partial} x_1^\eta} + \frac{\underline{\partial} g_{ij2}}{\underline{\partial} x_2^\eta}\right)\right]\left[y_1^i y_2^j + y_2^i y_1^j\right],
\end{aligned}$$

which are the complexified geodesics equations.

One more illustration concerns the Wong's equations which arise in the dynamics of a colored particle in Yang–Mills field and on the falling cat theorem [45]. The complexified Lagrangian and Hamiltonian of the theory on the complex Lie algebroid E are given by

$$
\begin{aligned}
&L(x^\eta(z), D_{(z)}^{(\alpha)} x^\eta(z), \overline{\mathrm{v}}^i(z)) = \frac{1}{2} h_{ij} \overline{\mathrm{v}}^i \overline{\mathrm{v}}^j + \frac{1}{2} g_{\eta\sigma} u^\eta u^\sigma \\
&= \frac{1}{2}(h_{ij}^1(\overline{\mathrm{v}}_1^i \overline{\mathrm{v}}_1^j - \overline{\mathrm{v}}_2^i \overline{\mathrm{v}}_2^j) - h_{ij}^2(\overline{\mathrm{v}}_1^i \overline{\mathrm{v}}_2^j + \overline{\mathrm{v}}_2^i \overline{\mathrm{v}}_1^j)) \\
&\quad + i[h_{ij}^1(\overline{\mathrm{v}}_1^i \overline{\mathrm{v}}_2^j + \overline{\mathrm{v}}_2^i \overline{\mathrm{v}}_1^j) + h_{ij}^2(\overline{\mathrm{v}}_1^i \overline{\mathrm{v}}_1^j - \overline{\mathrm{v}}_2^i \overline{\mathrm{v}}_2^j)]) \\
&\quad + \frac{1}{2}(g_{\eta\sigma}^1(u_1^\eta u_1^\sigma - u_2^\eta u_2^\sigma) - g_{\eta\sigma}^2(u_1^\eta u_2^\sigma + u_2^\eta u_1^\sigma)) \\
&\quad + i[g_{\eta\sigma}^1(u_1^\eta u_2^\sigma + u_2^\eta u_1^\sigma) + g_{\eta\sigma}^2(u_1^\eta u_1^\sigma - u_2^\eta u_2^\sigma) \\
&= \underbrace{\frac{1}{2}(h_{ij}^1(\overline{\mathrm{v}}_1^i \overline{\mathrm{v}}_1^j - \overline{\mathrm{v}}_2^i \overline{\mathrm{v}}_2^j) - h_{ij}^2(\overline{\mathrm{v}}_1^i \overline{\mathrm{v}}_2^j + \overline{\mathrm{v}}_2^i \overline{\mathrm{v}}_1^j))}_{L_1(h(z))} \\
&\quad + \underbrace{\frac{1}{2}(g_{\eta\sigma}^1(u_1^\eta u_1^\sigma - u_2^\eta u_2^\sigma) - g_{\eta\sigma}^2(u_1^\eta u_2^\sigma + u_2^\eta u_1^\sigma))}_{L_1(g(z))} \\
&\quad + \frac{1}{2} i \left[\underbrace{h_{ij}^1(\overline{\mathrm{v}}_1^i \overline{\mathrm{v}}_2^j + \overline{\mathrm{v}}_2^i \overline{\mathrm{v}}_1^j) + h_{ij}^2(\overline{\mathrm{v}}_1^i \overline{\mathrm{v}}_1^j - \overline{\mathrm{v}}_2^i \overline{\mathrm{v}}_2^j)}_{L_2(h(z))} \right. \\
&\qquad \left. + \underbrace{g_{\eta\sigma}^1(u_1^\eta u_2^\sigma + u_2^\eta u_1^\sigma) + g_{\eta\sigma}^2(u_1^\eta u_1^\sigma - u_2^\eta u_2^\sigma)}_{L_2(g(z))} \right] \\
&= L_1(h(z)) + L_1(g(z)) + i[L_2(h(z)) + L_2(g(z))]
\end{aligned}
$$

for $x^\eta = x_1^\eta + i x_2^\eta$; $\overline{\mathrm{v}}^i = \overline{\mathrm{v}}_1^i + i \overline{\mathrm{v}}_2^i$; $u^\eta = u_1^\eta + i u_2^\eta$ and

$$
\begin{aligned}
&\mathbf{H}(x^\eta(z), p_\eta(z), \overline{p}_i(z)) = \frac{1}{2} h^{ij} \overline{p}_i \overline{p}_j + \frac{1}{2} g^{\eta\sigma} p_\eta p_\sigma, \\
&= \underbrace{\frac{1}{2}(h_1^{ij}(\overline{p}_i^1 \overline{p}_j^1 - \overline{p}_i^2 \overline{p}_j^2) - h_2^{ij}(\overline{p}_i^1 \overline{p}_j^2 + \overline{p}_i^2 \overline{p}_j^1))}_{H_1(h(z))} \\
&\quad + \underbrace{\frac{1}{2}(g_1^{\eta\sigma}(p_\eta^1 p_\sigma^1 - p_\eta^2 p_\sigma^2) - g_2^{\eta\sigma}(p_\eta^1 p_\sigma^2 + p_\eta^2 p_\sigma^1))}_{H_1(g(z))}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}i\left[\underbrace{h_2^{ij}(\overline{p}_i^1\overline{p}_j^2+\overline{p}_i^2\overline{p}_j^1)+h_2^{ij}(\overline{p}_i^1\overline{p}_j^1-\overline{p}_i^2\overline{p}_j^2)}_{H_2(h(z))}\right.\\
&\qquad\left.+\underbrace{g_1^{\eta\sigma}(p_\eta^1p_\sigma^2+p_\eta^2p_\sigma^1)+g_2^{\eta\sigma}(p_\eta^1p_\sigma^1-p_\eta^2p_\sigma^2)}_{H_2(g(z))}\right]\\
&=\mathbf{H}_1(h(z))+\mathbf{H}_1(g(z))+i\left[\mathbf{H}_2(h(z))+\mathbf{H}_2(g(z))\right]
\end{aligned}
$$

for $x^\eta = x_1^\eta + ix_2^\eta$ and $\overline{p}^i = \overline{p}_1^i + i\overline{p}_2^i$, where $u = \dot{x}$,

$$(x^\eta, \dot{x}^\eta, \overline{\mathrm{v}}^i) \equiv (x_1^\eta + ix_2^\eta, \dot{x}_1^\eta + i\dot{x}_2^\eta, \overline{\mathrm{v}}_1^i + i\overline{\mathrm{v}}_2^i),$$

is the analogous dual fibered complexified coordinates on TQ/G and

$$(x^\eta, p_\eta, \overline{p}_i) \equiv (x_1^\eta + ix_2^\eta, p_\eta^1 + ip_\eta^2, \overline{p}_i^1 + i\overline{p}_i^2),$$

is the dual complexified coordinates on T^*Q/G, G being a compact Lie group, $E = TM \times \mathcal{A}$ with $\dim TM = m$ and $\dim \mathcal{A} = n$, $\mathcal{A}$ being an arbitrary Lie $\mathbb{C}$-algebra of complex dimension n. h is a complex metric on $\mathcal{A}$ and g is assumed to be a Riemannian complex metric on M with $h = h_1 + ih_2$ and $g = g_1 + ig_2$. We may now derive the complexified Wong's equation on the complex plane which is split into the real and complex parts correspondingly:

$$
\begin{aligned}
\frac{\overline{d}\overline{\mathrm{v}}_1^i}{\overline{d}\zeta} = \overline{C}_{ji}^l &\Big[(h_1^{ik}h_{ls1}-h_2^{ik}h_{ls2})(\overline{\mathrm{v}}_1^j\overline{\mathrm{v}}_1^s-\overline{\mathrm{v}}_2^j\overline{\mathrm{v}}_2^s)\\
&-(h_1^{ik}h_{ls2}+h_2^{ik}h_{ls1})(\overline{\mathrm{v}}_1^j\overline{\mathrm{v}}_2^s+\overline{\mathrm{v}}_2^j\overline{\mathrm{v}}_1^s)\Big]\\
&+\underline{C}_{ji}^l\Big[(h_1^{ik}h_{ls1}-h_2^{ik}h_{ls2})(\overline{\mathrm{v}}_1^j\overline{\mathrm{v}}_2^s+\overline{\mathrm{v}}_2^j\overline{\mathrm{v}}_1^s)\\
&+(h_1^{ik}h_{ls2}+h_2^{ik}h_{ls1})(\overline{\mathrm{v}}_1^j\overline{\mathrm{v}}_1^s-\overline{\mathrm{v}}_2^j\overline{\mathrm{v}}_2^s)\Big]+(\alpha-1)\Big[p\overline{\mathrm{v}}_1^k-q\overline{\mathrm{v}}_2^k\Big],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\underline{d}\overline{\mathrm{v}}_2^i}{\underline{d}\zeta} = \overline{C}_{ji}^l &\Big[(h_1^{ik}h_{ls1}-h_2^{ik}h_{ls2})(\overline{\mathrm{v}}_1^j\overline{\mathrm{v}}_2^s+\overline{\mathrm{v}}_2^j\overline{\mathrm{v}}_1^s)\\
&+(h_1^{ik}h_{ls2}+h_2^{ik}h_{ls1})(\overline{\mathrm{v}}_1^j\overline{\mathrm{v}}_1^s-\overline{\mathrm{v}}_2^j\overline{\mathrm{v}}_2^s)\Big]\\
&-\underline{C}_{ji}^l\Big[(h_1^{ik}h_{ls1}-h_2^{ik}h_{ls2})(\overline{\mathrm{v}}_1^j\overline{\mathrm{v}}_1^s-\overline{\mathrm{v}}_2^j\overline{\mathrm{v}}_2^s)\\
&-(h_1^{ik}h_{ls2}+h_2^{ik}h_{ls1})(\overline{\mathrm{v}}_1^j\overline{\mathrm{v}}_2^s+\overline{\mathrm{v}}_2^j\overline{\mathrm{v}}_1^s)\Big]+(\alpha-1)\Big[p\overline{\mathrm{v}}_1^k+q\overline{\mathrm{v}}_2^k\Big].
\end{aligned}
$$

A number of applications of the complexified geodesics equations and complexified Wong's equations within the framework of quantum field theory are under advancement.

3. Complexified hamiltonian for complex Lie algebroids. Let $L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right)$ be a hyperregular complexified Lagrangian on the complex Lie algebroid E with admissible curve q in E and with two fixed endpoints

$$A, B \in M \subset \mathcal{P}(J,E)_A^B = \{\mathcal{P}(J,E) \mid \pi(q(0)) = A \text{ and } \pi(q(z)) = B\},$$

and

$$\mathcal{H}\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right) = p_i y^i - L\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right),$$

where

$$p_i = \frac{\partial\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right)}{\partial y^i},$$

be the corresponding complexified Hamiltonian. If $(q, \dot{q})$ is the corresponding minimizer, then there exists p such that the generalized complexified Hamiltonian system for complex Lie algebroids:

$$\begin{aligned}
\frac{dx^\alpha(\zeta)}{d\zeta} &= \rho_i^\eta \frac{\partial \mathcal{H}}{\partial p_i}\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right), \\
\frac{dp_i(\zeta)}{d\zeta} &= -\rho_i^\eta \frac{\partial \mathcal{H}}{\partial x^\eta}\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right) \\
&\quad - \frac{(\alpha-1)\sinh\zeta}{\cosh z - \cosh\zeta} p_i - C_{i,j}^k p_k(\zeta) \frac{\partial \mathcal{H}}{\partial p_j}\left(\zeta, q(\zeta), D_{(z)}^{(\alpha)} q(\zeta)\right).
\end{aligned}$$

Their complexified parts are accordingly:

$$\begin{aligned}
\frac{\overline{d}x_1^\alpha}{\overline{d}\zeta_1} + \frac{\underline{d}x_2^\alpha}{\underline{d}\zeta_2} &= 2\rho_i^\eta \left[\frac{\overline{\partial}\mathcal{H}_1}{\overline{\partial}p_j^1} + \frac{\underline{\partial}\mathcal{H}_2}{\underline{\partial}p_j^2}\right], \\
\frac{\overline{d}x_2^\alpha}{\overline{d}\zeta_1} - \frac{\underline{d}x_1^\alpha}{\underline{d}\zeta_2} &= 2\rho_i^\eta \left[\frac{\overline{\partial}\mathcal{H}_2}{\overline{\partial}p_j^1} - \frac{\underline{\partial}\mathcal{H}_1}{\underline{\partial}p_j^2}\right],
\end{aligned}$$

$$\begin{aligned}
\frac{\overline{d}p_{i,1}}{\overline{d}\zeta_1} + \frac{\underline{d}p_{i,2}}{\underline{d}\zeta_2} &= -\frac{1}{2}\rho_i^\eta \frac{\partial \mathcal{H}_1}{\partial x_1^\eta} - (\alpha-1)(pp_{i,1} - qp_{i,2}) \\
&\quad - \frac{1}{2}\left[\overline{C}_{i,j}^k \left(p_{k,1}\frac{\partial \mathcal{H}_1}{\partial p_{j,1}} + p_{k,2}\frac{\partial \mathcal{H}_2}{\partial p_{j,2}}\right) + \underline{C}_{i,j}^k \left(p_{k,2}\frac{\partial \mathcal{H}_1}{\partial p_{j,1}} - p_{k,1}\frac{\partial \mathcal{H}_2}{\partial p_{j,2}}\right)\right], \\
\frac{\overline{d}p_{i,1}}{\overline{d}\zeta_1} - \frac{\underline{d}p_{i,1}}{\underline{d}\zeta_2} &= \frac{1}{2}\rho_i^\eta \frac{\partial \mathcal{H}_2}{\partial x_2^\eta} - (\alpha-1)(pp_{i,2} + qp_{i,1}) \\
&\quad - \frac{1}{2}\left[\overline{C}_{i,j}^k \left(p_{k,2}\frac{\partial \mathcal{H}_1}{\partial p_{j,1}} - p_{k,1}\frac{\partial \mathcal{H}_2}{\partial p_{j,2}}\right) - \underline{C}_{i,j}^k \left(p_{k,1}\frac{\partial \mathcal{H}_1}{\partial p_{j,1}} + p_{k,2}\frac{\partial \mathcal{H}_2}{\partial p_{j,2}}\right)\right].
\end{aligned}$$

Lemma 3.1. *The fractional complexified Hamiltonian dynamics on the complex dual bundle E^* is represented by the complex vector fields:*

$$
\begin{aligned}
&\mathcal{D}^r(z,\zeta) = \rho_i^\eta \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial}{\partial x^\eta} - (\alpha-1)(p+iq)\frac{\partial}{\partial \zeta_i} - \left(\rho_i^\eta \frac{\partial \mathcal{H}}{\partial x^\eta} + C_{i,j}^k p_k(t)\frac{\partial \mathcal{H}}{\partial p_j}\right)\frac{\partial}{\partial \zeta_i}\\
&= \frac{1}{2}\rho_i^\eta\left[\left(\frac{\overline{\partial}\mathcal{H}_1}{\overline{\partial}p_j^1} + \frac{\underline{\partial}\mathcal{H}_2}{\underline{\partial}p_j^2}\right)\frac{\partial}{\partial x_1^\eta} + \left(\frac{\overline{\partial}\mathcal{H}_2}{\overline{\partial}p_j^1} - \frac{\underline{\partial}\mathcal{H}_1}{\underline{\partial}p_j^2}\right)\frac{\partial}{\partial x_2^\eta}\right]\\
&\quad - \frac{1}{2}(\alpha-1)\left[p\frac{\partial}{\partial \zeta_{i,1}} + q\frac{\partial}{\partial \zeta_{i,2}}\right]\\
&\quad - \frac{1}{2}\rho_i^\eta\left(\frac{\overline{\partial}\mathcal{H}_1}{\overline{\partial}x_1^\eta} + \frac{\underline{\partial}\mathcal{H}_2}{\underline{\partial}x_2^\eta}\right)\frac{\partial}{\partial \zeta_1} + \frac{1}{2}\rho_i^\eta\left(\frac{\overline{\partial}\mathcal{H}_2}{\overline{\partial}x_1^\eta} - \frac{\underline{\partial}\mathcal{H}_1}{\underline{\partial}x_2^\eta}\right)\frac{\partial}{\partial \zeta_2}\\
&\quad - \frac{1}{2}\left(\overline{C}_{i,j}^k p_{k,1} + \underline{C}_{i,j}^k p_{k,2}\right)\left[\left(\frac{\overline{\partial}\mathcal{H}_1}{\overline{\partial}p_j^1} + \frac{\underline{\partial}\mathcal{H}_2}{\underline{\partial}p_j^2}\right)\frac{\partial}{\partial \zeta_1} + \left(\frac{\overline{\partial}\mathcal{H}_2}{\overline{\partial}p_j^1} - \frac{\underline{\partial}\mathcal{H}_1}{\underline{\partial}p_j^2}\right)\frac{\partial}{\partial \zeta_2}\right]\\
&\quad + \frac{1}{2}\left(\overline{C}_{i,j}^k p_{k,2} - \underline{C}_{i,j}^k p_{k,1}\right)\left[\left(\frac{\overline{\partial}\mathcal{H}_2}{\overline{\partial}p_j^1} - \frac{\underline{\partial}\mathcal{H}_1}{\underline{\partial}p_j^2}\right)\frac{\partial}{\partial \zeta_1} - \left(\frac{\overline{\partial}\mathcal{H}_1}{\overline{\partial}p_j^1} + \frac{\underline{\partial}\mathcal{H}_2}{\underline{\partial}p_j^2}\right)\frac{\partial}{\partial \zeta_2}\right]\\
&\quad + i\Bigg\{\frac{1}{2}\rho_i^\eta\left[\left(\frac{\overline{\partial}\mathcal{H}_2}{\overline{\partial}p_j^1} - \frac{\underline{\partial}\mathcal{H}_1}{\underline{\partial}p_j^2}\right)\frac{\partial}{\partial x_1^\eta} - \left(\frac{\overline{\partial}\mathcal{H}_1}{\overline{\partial}p_j^1} + \frac{\underline{\partial}\mathcal{H}_2}{\underline{\partial}p_j^2}\right)\frac{\partial}{\partial x_2^\eta}\right]\\
&\quad - \frac{1}{2}(\alpha-1)\left(q\frac{\partial}{\partial \zeta_{i,1}} - p\frac{\partial}{\partial \zeta_{i,2}}\right)\\
&\quad - \frac{1}{2}\rho_i^\eta\left(\frac{\overline{\partial}\mathcal{H}_2}{\overline{\partial}x_1^\eta} - \frac{\underline{\partial}\mathcal{H}_1}{\underline{\partial}x_2^\eta}\right)\frac{\partial}{\partial \zeta_1} + \frac{1}{2}\rho_i^\eta\left(\frac{\overline{\partial}\mathcal{H}_1}{\overline{\partial}x_1^\eta} + \frac{\underline{\partial}\mathcal{H}_2}{\underline{\partial}x_2^\eta}\right)\frac{\partial}{\partial \zeta_2}\\
&\quad - \frac{1}{2}\left(\overline{C}_{i,j}^k p_{k,1} + \underline{C}_{i,j}^k p_{k,2}\right)\left[\left(\frac{\overline{\partial}\mathcal{H}_2}{\overline{\partial}p_j^1} - \frac{\underline{\partial}\mathcal{H}_1}{\underline{\partial}p_j^2}\right)\frac{\partial}{\partial \zeta_1} - \left(\frac{\overline{\partial}\mathcal{H}_1}{\overline{\partial}p_j^1} + \frac{\underline{\partial}\mathcal{H}_2}{\underline{\partial}p_j^2}\right)\frac{\partial}{\partial \zeta_2}\right]\\
&\quad - \frac{1}{2}\left(\overline{C}_{i,j}^k p_{k,1} - \underline{C}_{i,j}^k p_{k,2}\right)\left[\left(\frac{\overline{\partial}\mathcal{H}_1}{\overline{\partial}p_j^1} + \frac{\underline{\partial}\mathcal{H}_2}{\underline{\partial}p_j^2}\right)\frac{\partial}{\partial \zeta_1} + \left(\frac{\overline{\partial}\mathcal{H}_2}{\overline{\partial}p_j^1} - \frac{\underline{\partial}\mathcal{H}_1}{\underline{\partial}p_j^2}\right)\frac{\partial}{\partial \zeta_2}\right]\Bigg\}\\
&= \Re\mathcal{D}^r(z,\zeta) + i\Im\mathcal{D}^r(z,\zeta) \equiv \overline{\mathcal{D}}^r(z,\zeta) + i\underline{\mathcal{D}}^r(z,\zeta).
\end{aligned}
$$

4. Conclusions and perspectives. In our opinion, this work represents the first attempt to construct the theoretical framework of complex Lie algebroids based on extended fractional variational formalism. That will be the foundation, we believe however, of more stimulating investigations. The complexified fractional Euler–Lagrange equations, the complexified Hamilton equations and the complexified geodesics and Wong's equations for complex Lie algebroids are derived. It is the author's speculations that they could have motivating outcomes on both geometric dynamics and quantum field theory [11, 13, 14, 15, 22]. The new class of complexified Lagrangian and Hamiltonian systems obtained on complex Lie algebroids is wider than the standard class of Lagrangian and Hamiltonian dynamical systems. Decomplexification of the dynamical equations, in particular the fractional Euler–Lagrange equations, geodesic equations and Wong's equations could be realized if the action is complexified in the theory. The occurrence of the new complexified action in the theory could have considerable consequences in quantum field theory, where the complex part of the fractional action may guide to a novel complexified dynamics which may well differ entirely from the classical mechanics cardinally. From a phenomenological standpoint, we can go further and construct a fractional complexified Hamiltonian approach to holomorphic Poisson manifolds and holomorphic complex Lie algebroids from the viewpoint of real Poisson geometry. We expect they will open up in the future a new inspiring research area in diverse branches of physics and supply us with a powerful tool to figure out many basic problems in the area of complexified dynamical systems and noncommutative geometry. Future research efforts may be directed towards formulating predictions that can be tracked and tested numerically. The results in this work declare additional oversimplification and advocate a supplementary study.

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