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On a theorem of Lindelöf

Dedicated to the memory of Professor Jan G. Krzyż

Abstract. We give a quasiconformal version of the proof for the classical Lindelöf theorem: Let f map the unit disk $\mathbb{D}$ conformally onto the inner domain of a Jordan curve $\mathcal{C}$. Then $\mathcal{C}$ is smooth if and only if $\arg f'(z)$ has a continuous extension to $\overline{\mathbb{D}}$. Our proof does not use the Poisson integral representation of harmonic functions in the unit disk.

1. Introduction. Let $f : \mathbb{D} \to \mathbb{C}$ be a conformal mapping of the unit disk $\mathbb{D}$ onto $f(\mathbb{D})$. The smoothness of $\partial f(\mathbb{D})$ yields the smoothness of f on $\partial\mathbb{D}$. The classical Lindelöf theorem [7] as well as Warschawski's theorem [9] on differentiability of f at the boundary $\partial\mathbb{D}$ are the basic results of this kind of behavior.

In this paper we adopt a different point of view. Assuming that the boundary curve is smooth, i.e. it has a continuously turning tangent, we extend f over the unit disk to a quasiconformal mapping and apply some results from the infinitesimal geometry of quasiconformal mappings developed in [5], see also [4]. In order to illustrate our approach, we give a quasiconformal version of the proof for the aforementioned Lindelöf theorem. Recall that the standard proof of the Lindelöf theorem is based on the

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Poisson formula, see, e.g. [8], p. 44. Our version of the proof does not use the Poisson integral representation of harmonic functions in the unit disk $\mathbb{D}$. In order to make our method easily understandable, we have collected in Chapter 3 basic notations and auxiliary lemmas from the geometric theory of plane quasiconformal mappings.

2. The Lindelöf Theorem. Let f map $\mathbb{D}$ conformally onto the inner domain of a smooth Jordan curve $\mathcal{C}$. Since the characterization of smoothness in terms of tangent does not depend on the parametrization, we may choose the *conformal parametrization*

$$\mathcal{C}\,:\,w(t)=f(e^{it}),\quad 0\le t\le 2\pi.$$

An analytic characterization of the smoothness is given by the classical Lindelöf [7] theorem:

Theorem 1. *Let f map $\mathbb{D}$ conformally onto the inner domain of a Jordan curve $\mathcal{C}$. Then $\mathcal{C}$ is smooth if and only if $\arg f'(z)$ has a continuous extension to $\overline{\mathbb{D}}$. If $\mathcal{C}$ is smooth, then*

$$\arg f'(e^{it})=\beta(t)-t-\frac{\pi}{2},\tag{2.1}$$

where $\beta(t)$ stands for the tangent angle of the curve $f(e^{it})$ at the point t.

Proof. Let $\mathcal{C}$ be a closed smooth Jordan curve in the complex plane $\mathbb{C}$ and f be a conformal mapping of the disk $\mathbb{D}$ onto the inner domain of $\mathcal{C}$. The smoothness of $\mathcal{C}$ implies the existence of a continuous function $\beta(t)$ on the segment $[0,2\pi]$ such that

$$\arg\left[f(e^{i\theta})-f(e^{it})\right]\to\begin{cases}\beta(t), & \text{as } \theta\to t+0,\\ \beta(t)+\pi, & \text{as } \theta\to t-0.\end{cases}$$

Since each smooth curve $\mathcal{C}$ is asymptotically conformal, see [8], p. 246, the mapping f can be extended to a quasiconformal mapping of the complex plane $\mathbb{C}$ in such a way that the corresponding complex dilatation $\mu(z)$ will satisfy the condition $\mu(z)\to 0$ as $|z|\to 1+$. On the other hand, the standard rescaling arguments and convergence and compactness theory imply, see Lemma 1, that for the extended mapping

$$\lim_{z,\zeta\to 0}\left\{\frac{f(z+\eta)-f(\eta)}{f(\zeta+\eta)-f(\eta)}-\frac{z}{\zeta}\right\}=0\tag{2.2}$$

uniformly with respect to $\eta\in\partial\mathbb{D}$, provided that $|z/\zeta|\le\delta$ for each fixed $\delta>0$. If we replace z by $z\zeta$, then we get that

$$\lim_{\zeta\to 0}\frac{f(\zeta z+\eta)-f(\eta)}{f(\zeta+\eta)-f(\eta)}=z\tag{2.3}$$

locally uniformly in $z \in \mathbb{C}$ and uniformly in $\eta \in \partial\mathbb{D}$. In particular, setting $\zeta = re^{i\theta_1}$ and $z = re^{i(\theta_2-\theta_1)}$, we obtain

$$\lim_{r\to 0}\left[\arg\frac{f(\eta+re^{i\theta_2})-f(\eta)}{re^{i\theta_2}} - \arg\frac{f(\eta+re^{i\theta_1})-f(\eta)}{re^{i\theta_1}}\right] = 0 \tag{2.4}$$

for an appropriate branch of the argument uniformly in $\theta_1, \theta_2 \in [0, 2\pi]$ and $\eta \in \partial\mathbb{D}$. Let Γ be an arc of the unit circle $\partial\mathbb{D}$ ending at the point $\eta = e^{it}$. Since

$$\lim_{\substack{z\to e^{it}\\ z\in\Gamma}}\left[\arg\frac{f(z)-f(e^{it})}{z-e^{it}}\right] = \beta(t) - t - \frac{\pi}{2},$$

we see that the relation (2.4) implies the existence of the limit

$$\arg f'(e^{it}) = \lim_{\substack{z\to e^{it}\\ z\in\overline{\mathbb{D}}}}\arg\frac{f(z)-f(e^{it})}{z-e^{it}} = \beta(t) - t - \frac{\pi}{2} \tag{2.5}$$

which is uniform with respect to the parameter t.

In order to prove that $\arg f'(z)$ has a continuous extension to the closed unit disk we proceed as follows.

For $z = 1+\rho e^{i\theta}$ in the disk $|z-1| < 1$, i.e. $\rho < 1$, we have $|(r-1)z+1| = |r-1+\rho e^{i\theta}(r-1)+1| < r+(1-r)\rho < 1$, i.e. $\eta(r-1)z+\eta \in \mathbb{D}$ for $\eta \in \partial\mathbb{D}$. Since f is analytic in $\mathbb{D}$, the functions of the family

$$F_r(z) = \frac{f(\eta(r-1)z+\eta)-f(\eta)}{f(r\eta)-f(\eta)}$$

are analytic at the point $z = 1$ for each $0 < r < 1$. Since $F_r(z) \to z$ as $r \to 1-0$ locally uniformly in $z \in \mathbb{D}$, the Weierstrass theorem yields that ${F_r}'(1) \to 1$, i.e.

$$\lim_{r\to 1-0}\frac{f'(r\eta)(r\eta-\eta)}{f(r\eta)-f(\eta)} = 1 \tag{2.6}$$

uniformly in $\eta \in \partial\mathbb{D}$. Formula (2.6) is the well-known Visser–Ostrowski condition, see, e.g. [8], p. 252.

Thus,

$$\lim_{r\to 1-0}\left(\arg f'(re^{it}) - \arg\frac{f(re^{it})-f(e^{it})}{re^{it}-e^{it}}\right) = 0$$

uniformly in t. Hence, by (2.5), there exists the limit

$$\lim_{r\to 1-0}\arg f'(re^{it}) = \arg f'(e^{it}) = \beta(t) - t - \frac{\pi}{2}$$

which is uniform in $t \in [0, 2\pi]$. The latter formula and the continuity of $\arg f'(e^{it})$ on $\partial\mathbb{D}$ implies the required continuous extension of $\arg f'(z)$ to $\overline{\mathbb{D}}$. Thus, we complete the proof of the first part of the theorem.

The converse part of the theorem is elementary and we refer the reader to the standard text given in [8], p. 44. □

3. On the infinitesimal geometry of QC-maps. This chapter contains some basic notions and auxiliary lemmas from geometric theory of plane quasiconformal mappings. These were used in our proof of the Lindelöf theorem.

Let G be a domain in the complex plane $\mathbb{C}$ and $\mu : G \to \mathbb{C}$ be a measurable function satisfying

$$\|\mu\|_\infty = \operatorname*{ess\,sup}_G |\mu(z)| < 1. \tag{3.1}$$

An orientation preserving homeomorphism $f : G \to \mathbb{C}$ of the Sobolev class $W^{1,2}_{\text{loc}}$ is called *quasiconformal* with complex dilatation μ, if it satisfies the Beltrami equation

$$f_{\bar z} = \mu(z) f_z \quad a.e. \tag{3.2}$$

A Jordan curve $\Gamma \subset \mathbb{C}$ is called a *quasiconformal curve* or *quasicircle* if it is the image of the unit circle under a quasiconformal mapping of $\mathbb{C}$, see, e.g. [8], p. 107. In 1963 L. Ahlfors [1] gave a simple geometric characterization of quasicircles. He proved that the curve Γ is a quasicircle iff the quantity

$$\gamma \equiv \gamma(w_1, w_2, w) = \frac{|w_1 - w| + |w - w_2|}{|w_1 - w_2|} \tag{3.3}$$

is bounded for all $w_1, w_2 \in \Gamma$ and $w \in \Gamma(w_1, w_2)$, where $\Gamma(w_1, w_2)$ denotes the sub-arc of Γ corresponding to $w_1, w_2 \in \Gamma$ with smaller diameter.

Let $\Gamma \subset \mathbb{C}$ be a quasicircle in the complex plane and let f denote a conformal mapping of the unit disk $\mathbb{D} = \{z : |z| < 1\}$ onto the interior of Γ. By a result of L. Ahlfors, see [2], p. 71, f admits a quasiconformal extension over the unit circle $\partial\mathbb{D}$. If there exists a quasiconformal extension with complex dilatation $\mu(z)$ such that

$$\operatorname*{ess\,sup}_{1 \le |z| \le t} |\mu(z)| \to 0, \quad t \to 1 + 0, \tag{3.4}$$

then the curve Γ is called *asymptotically conformal*, see [8], p. 246.

Ch. Pommerenke and J. Becker proved, see [8], p. 247, that (3.4) is equivalent to the condition

$$\lim_{|w_1 - w_2| \to 0} \frac{|w_1 - w| + |w - w_2|}{|w_1 - w_2|} = 1 \tag{3.5}$$

uniformly with respect to $w \in \Gamma(w_1, w_2)$.

It is easy to see that every smooth closed Jordan curve $\Gamma \subset \mathbb{C}$ is asymptotically conformal.

The following result is a key lemma on infinitesimal behavior on the boundary for quasiconformal extensions of conformal mappings. Its proof has been given in [3], see also [4].

Lemma 1. *Let f be a conformal mapping of $\mathbb{D}$ onto the interior of a Jordan domain $G \subset \mathbb{C}$ bounded by an asymptotically conformal (in particular,* *smooth) curve* $\Gamma = \partial G$. *Then* f *can be extended quasiconformally to* $\mathbb{C}$ *in such a way that*

$$\lim_{z,\zeta\to 0}\left\{\frac{f(z+\eta)-f(\eta)}{f(\zeta+\eta)-f(\eta)}-\frac{z}{\zeta}\right\}=0 \tag{3.6}$$

uniformly with respect to $\eta \in \partial\mathbb{D}$, *provided that* $|z/\zeta| \le \delta$ *for each fixed* $\delta > 0$.

Proof. Since $\partial G = f(\partial\mathbb{D})$ is asymptotically conformal, there exists a quasiconformal extension of f over the unit disk $\mathbb{D}$ to $\mathbb{C}$ with complex dilatation $\mu(z)$ such that

$$\operatorname*{ess\,sup}_{1<|z|\le 1+t} |\mu(z)| \to 0, \quad t \to +0. \tag{3.7}$$

For the extended mapping f, let us consider the following approximating family of f at $\eta \in \partial\mathbb{D}$, see [5],

$$F_{t,\eta}(z) = \frac{f(tz+\eta)-f(\eta)}{f(t+\eta)-f(\eta)}, \quad t > 0.$$

We shall consider the class $\mathfrak{F}_Q$ of all Q-quasiconformal self-mappings of the extended complex plane normalized with the conditions $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$. Note that this space of quasiconformal mappings is sequentially compact with respect to the locally uniform convergence, see [6], p. 73.

Now we see that all the mappings $F_{t,\eta}$ are in the class $\mathfrak{F}_Q$, $(t,\eta) \in \mathbb{R}^+ \times \partial\mathbb{D}$. Since $\mathfrak{F}_Q$ is sequentially compact, every convergent subsequence F_{t_n,η_n} as $n \to \infty$ has a limit mapping F_0 which is in the class $\mathfrak{F}_Q$.

Suppose that (3.6) does not hold. Then we can find $\varepsilon > 0$ and sequences $z_n \to 0$, $\zeta_n \to 0$ as $n \to \infty$, satisfying the inequality $|z_n/\zeta_n| \le \lambda$ for some $\lambda > 0$, and $\eta_n \in \partial\mathbb{D}$ such that

$$\left|\frac{f(z_n+\eta_n)-f(\eta_n)}{f(\zeta_n+\eta_n)-f(\eta_n)}-\frac{z_n}{\zeta_n}\right| > \varepsilon. \tag{3.8}$$

We write $F_n = F_{|\zeta_n|,\eta_n}$. All the functions F_n, $n = 1, 2, \dots$, belong to the space $\mathfrak{F}_Q$ and have complex dilatations $\mu_{F_n}(z) = \mu(|\zeta_n|z+\eta_n)$. From (3.7) it follows that $\mu(|\zeta_n|z+\eta_n) \to 0$ as $n \to \infty$ almost everywhere in $\mathbb{C}$. Without loss of generality we may assume that F_n converges locally uniformly in $\mathbb{C}$ to a quasiconformal mapping $F_0 \in \mathfrak{F}_Q$ and simultaneously that the sequence of their complex dilatations μ_{F_n} converges to 0 almost everywhere in $\mathbb{C}$ as $n \to \infty$. Otherwise, one can pass to an appropriate subsequence.

Next, we apply the well-known Bers–Bojarski convergence theorem. This theorem states that if f_n is a sequence of K-quasiconformal mappings of G which converges locally uniformly to a quasiconformal mapping f with complex dilatation μ_f, and if their complex dilatations μ_n tend to a limit function μ a.e. in G, then $\mu = \mu_f$ a.e. in G, see [6], p. 187–188. Thus, the limit function F_0 must have the complex dilatation $\mu_0 \equiv 0$. Applying the measurable Riemann mapping theorem, see [6], p. 194, we see that $F_0(z) = z$.

Let now the sequences z_n and ζ_n be chosen in such a way that $z_n/|\zeta_n| \to z_0 \in \mathbb{C}$. Since the unit circle is compact one can also assume that $\zeta_n/|\zeta_n| \to \zeta_0$, $|\zeta_0| = 1$. Hence

$$\lim_{n\to\infty} \left| \frac{f(z_n+\eta_n)-f(\eta_n)}{f(\zeta_n+\eta_n)-f(\eta_n)} - \frac{z_n}{\zeta_n} \right| = \lim_{n\to\infty} \left| \frac{F_n(z_n/|\zeta_n|)}{F_n(\zeta/|\zeta_n|)} - \frac{z_n}{\zeta_n} \right|$$
$$= \lim_{n\to\infty} \left| \frac{F_n(z_n/|\zeta_n|)}{F_n(\zeta/|\zeta_n|)} - \frac{z_n/|\zeta_n|}{\zeta_n/|\zeta_n|} \right| = \left| \frac{z_0}{\zeta_0} - \frac{z_0}{\zeta_0} \right| = 0$$

which contradicts (3.8). □

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