doi: 10.2478/v10062-011-0014-5

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXV, NO. 2, 2011 SECTIO A 63–73

JULIAN ŁAWRYNOWICZ and MASSIMO VACCARO

Structure fractals and para-quaternionic geometry

Professor Jan Krzyż in memoriam

Abstract. It is well known that starting with real structure, the Cayley–Dickson process gives complex, quaternionic, and octonionic (Cayley) structures related to the Adolf Hurwitz composition formula for dimensions $p = 2, 4$ and 8, respectively, but the procedure fails for $p = 16$ in the sense that the composition formula involves no more a triple of quadratic forms of the same dimension; the other two dimensions are $n = 2^7$. Instead, Ławrynowicz and Suzuki (2001) have considered graded fractal bundles of the flower type related to complex and Pauli structures and, in relation to the iteration process $p \to p + 2 \to p + 4 \to \dots$, they have constructed 2^4-dimensional "bipetals" for $p = 9$ and 2^7-dimensional "bisepals" for $p = 13$. The objects constructed appear to have an interesting property of periodicity related to the grading function on the fractal diagonal interpreted as the "pistil" and a family of pairs of segments parallel to the diagonal and equidistant from it, interpreted as the "stamens". The first named author, M. Nowak-Kępczyk, and S. Marchiafava (2006, 2009a, b) gave an effective, explicit determination of the periods and expressed them in terms of complex and quaternionic structures, thus showing the quaternionic background of that periodicity. In contrast to earlier results, the fractal bundle flower structure, in particular petals, sepals, pistils, and stamens are not introduced *ab initio*; they are quoted *a posteriori*, when they are fully motivated. Physical concepts of dual and conjugate objects as well as of antiparticles led us to extend the periodicity theorem to structure fractals in *para-quaternionic* formulation, applying some results in this direction by the second named author. The paper is concluded by outlining some applications.

2000 *Mathematics Subject Classification.* 81R25, 32L25, 53A50, 15A66.
Key words and phrases. Fractal, quaternion, para-quaternion, Clifford structure, para--quaternionic structure, bilinear form, quadratic form.

1. Introduction and statement of the periodicity theorem. Given generators $A_1^1, A_2^1, \dots, A_{2p-1}^1$ of a Clifford algebra $Cl_{2p-1}(\mathbb{C})$, $p = 2, 3, \dots$, in particular the generators

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of the Pauli algebra, consider the sequence

$$\begin{aligned} &A_\alpha^{q+1} = \sigma_3 \otimes i^\alpha A_\alpha^q \equiv \begin{pmatrix} i^\alpha A_\alpha^q & 0 \\ 0 & -i^\alpha A_\alpha^q \end{pmatrix}, \quad \alpha = 1, 2, \dots, 2p + 2q - 3; \\ (1)\quad &A_{2p+2q-2}^{q+1} = \sigma_1 \otimes I_{p,q} \equiv \begin{pmatrix} 0 & I_{p,q} \\ I_{p,q} & 0 \end{pmatrix}, \\ &A_{2p+2q-1}^{q+1} = -\sigma_2 \otimes I_{p,q} \equiv \begin{pmatrix} 0 & iI_{p,q} \\ -iI_{p,q} & 0 \end{pmatrix}, \end{aligned}$$

of generators of Clifford algebras $Cl_{2p+2q-1}(\mathbb{C})$, $q = 1, 2, \dots$, and the sequence of corresponding system of closed squares Q_q^α of diameter 1, centered at the origin of $\mathbb{C}$, where $I_{p,q} = I_{2^{p+q-2}}$, the unit matrix of order 2^{p+q-2}. It is convenient to start with q always from 1, i.e., to shift q for $\alpha \geq 2p$ correspondingly.

The difference in formula (1) here and in [4] is due to the replacement of matrix units $\mathbf{1}$, $i\sigma_1$, $i\sigma_2$, $i\sigma_3$ of the usual quaternions by the units

$$\mathbf{1}, \quad \mathbf{i} = i\sigma_2, \quad \mathbf{j} = \sigma_1, \quad \mathbf{k} = \sigma_3$$

of para-quaternions, so that our $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ mean $\mathbf{j}$, $(1/i)\mathbf{i}$, and $(1/i)\mathbf{k}$ in [4], respectively. This is due to our definition of the real Clifford algebra $\tilde{\mathbb{H}}$ of *para-quaternions* as generated by 1 and imaginary units i, j, k satisfying

$$-i^2 = j^2 = k^2 = 1, \quad ij = -ji = k.$$

For a para-quaternionic structure the left module structure is defined up to conjugation in $\tilde{\mathbb{H}}$.

Within a closed square Q_q^α consider its diameter

$$L_\infty = \left[\frac{1}{2\sqrt{2}}(-1+i);\ \frac{1}{2\sqrt{2}}(1-i)\right]$$

and two segments, symmetric and equidistant with respect to L_∞:

$$\begin{aligned} L_h^- &= \left[\frac{1}{2\sqrt{2}}(1+i-i\varepsilon_q^r);\ \frac{1}{2\sqrt{2}}(1-\varepsilon_q^r-i)\right], \\ L_h^+ &= \left[\frac{1}{2\sqrt{2}}(-1+\varepsilon_q^r+i);\ \frac{1}{2\sqrt{2}}(1-i+i\varepsilon_q^r)\right], \end{aligned}$$

where

$$\varepsilon_q^r = 1/2^h, \quad h = p + q - 1 - r, \quad r = \begin{cases} 2 & \text{for} \quad \alpha = 1, 2, \dots, 2p-1; \\ \left[\frac{1}{2}\alpha\right] & \text{for} \quad \alpha = 2p, 2p+1, \dots \end{cases}$$

and [ ] denotes the function "entier". Clearly, $\operatorname{dist}(L_h^{\pm}, L_\infty) = 1/2^{h+2}$.

Consider then the sets: L_∞^0 of points

$$z = \frac{1}{2\sqrt{2}}\frac{m}{2^n}(1-i), \quad m = 0, \pm 1, \dots, \pm(2^n - 1); \quad n = 0, 1, \dots,$$

of L_∞, and L_h^0 of points

$$z_-^-(h) = \frac{1}{2\sqrt{2}}\frac{m}{2^n}(1-i) - \frac{1}{2\sqrt{2}}\varepsilon_q^r \quad \text{and} \quad z_+^-(h) = \frac{1}{2\sqrt{2}}\frac{m}{2^n}(1-i) - \frac{i}{2\sqrt{2}}\varepsilon_q^r,$$

$m = 0, \pm 1, \dots, \pm(2^n - 1)$; $n = 0, 1, \dots$, of L_h^-,

$$z_+^+(h) = \frac{1}{2\sqrt{2}}\frac{m}{2^n}(1-i) + \frac{1}{2\sqrt{2}}\varepsilon_q^r \quad \text{and} \quad z_-^+(h) = \frac{1}{2\sqrt{2}}\frac{m}{2^n}(1-i) + \frac{i}{2\sqrt{2}}\varepsilon_q^r,$$

$m = 0, \pm 1, \dots, \pm(2^n - 1)$; $n = 0, 1, \dots$, of L_h^+.

Let

$$A_\alpha^q = (a_{\alpha j}^{qk}),\ A_\alpha = (a_{\alpha j}^k),\ j, k = 1, 2, \dots, 2^{p+q-2}. \tag{2}$$

Let further

$$g_q^\alpha(a_{\alpha j}^{qk}; z) = i^\alpha a_{\alpha j}^{qk} \text{ if } g_q^\alpha(z) = i^\alpha a_{\alpha j}^{qk}; \quad g_q^\alpha(a_{\alpha j}^{qk}; z) = 0 \text{ if } g_q^\alpha(z) \neq i^\alpha a_{\alpha j}^{qk},$$

where g_q^α is the gradating function equal $a_{\alpha j}^{qk}$ on the closed square $Q_{\alpha k}^{qj}$ corresponding to the pair (j,k); we suppose that the original square is divided into 4^{p+q-2} squares with sides parallel to the sides of Q_q^α.

Given $z \in L_\infty^0$, consider the sequences

$$g_1^\alpha(z),\ g_2^\alpha(z),\ \dots \quad \text{for } \alpha < 2p, \tag{3}$$

$$\hat g_1^\alpha(z_-^1),\ \hat g_2^\alpha(z_-^2),\ \dots \quad \text{for } \alpha < 2p, \tag{4}$$

$$\hat g_1^\alpha(z_+^1),\ \hat g_2^\alpha(z_+^2),\ \dots \quad \text{for } \alpha < 2p, \tag{5}$$

where

$$\begin{aligned} \hat g_q^\alpha(z_-^q) &= (g_q^\alpha(z_-^-(h)),\ g_q^\alpha(z_-^+(h)), \\ \hat g_q^\alpha(z_+^q) &= (g_q^\alpha(z_+^-(h)),\ g_q^\alpha(z_+^+(h)), \end{aligned} \tag{6}$$

as well as

$$\hat g_1^{2r}(z_-^1),\ \hat g_2^{2r}(z_-^2),\ \dots \quad \text{for } 2r = \alpha \geq 2p, \tag{7}$$

$$\hat g_1^{2r}(z_+^1),\ \hat g_2^{2r}(z_+^2),\ \dots \quad \text{for } 2r = \alpha \geq 2p, \tag{8}$$

$$\hat g_1^{2r+1}(z_-^1),\ \hat g_2^{2r+1}(z_-^2),\ \dots \quad \text{for } 2r+1 = \alpha > 2p, \tag{9}$$

$$\hat g_1^{2r+1}(z_+^1),\ \hat g_2^{2r+1}(z_+^2),\ \dots \quad \text{for } 2r+1 = \alpha > 2p, \tag{10}$$

with the notation (6), and

$$\hat{g}_1^{2r}(z_1^-),\, \hat{g}_2^{2r}(z_2^-),\, \dots \quad \text{for } 2r = \alpha \geq 2p, \tag{11}$$

$$\hat{g}_1^{2r}(z_1^+),\, \hat{g}_2^{2r}(z_2^+),\, \dots \quad \text{for } 2r = \alpha \geq 2p, \tag{12}$$

$$\hat{g}_1^{2r+1}(z_1^-),\, \hat{g}_2^{2r+1}(z_2^-),\, \dots \quad \text{for } 2r+1 = \alpha > 2p, \tag{13}$$

$$\hat{g}_1^{2r+1}(z_1^+),\, \hat{g}_2^{2r+1}(z_1^+),\, \dots \quad \text{for } 2r+1 = \alpha > 2p, \tag{14}$$

where

$$\hat{g}_q^\alpha(z_q^-) = (g_q^\alpha(z_-^-(h)),\, g_q^\alpha(z_+^-(h)), \quad \hat{g}_q^\alpha(z_q^+) = (g_q^\alpha(z_-^+(h)),\, g_q^\alpha(z_+^+(h)). \tag{15}$$

We need two lemmas.

Lemma 1 ([4]). *Formulae* (1) *are equivalent to*

$$\begin{aligned} &A_\alpha^{q=1} = i^\alpha \mathbf{k} \otimes A_\alpha^q, \quad \alpha = 1, 2, \dots, 2p+2q-3; \\ &A_{2p+2q-2}^{q+1} = \mathbf{j} \otimes \mathbf{1}^{\otimes(p+q-2)}, \quad A_{2p+2q-1}^{q+1} = i\mathbf{i} \otimes \mathbf{1}^{\otimes(p+q-2)}. \end{aligned} \tag{16}$$

Lemma 2 ([4]). (i) *If a quaternionic vector space has dimension $4n$, its para-quaternionic counterpart has dimension $2n$; $n \in \mathbb{N}$.*

(ii) *If* $\dim V = 4n$ *in both cases: quaternionic and para-quaternionic, there exists a basis of V of the following type:*

$$(X_1, \dots, X_n, J_1X_1, \dots, J_1X_n, J_2X_1, \dots, J_2X_n, J_3X_1, \dots, J_3X_n) \tag{17}$$

for any admissible basis (J_1, J_2, J_3).

We have

Periodicity Theorem (para-quaternionic formulation). (i) *If $a_{\alpha\lambda}^\lambda \neq 0$ for $\lambda = 2^{p+q-2}$, the sequences* (3) *are periodic of period* 2*, starting from some term. The periods are:*

$$\begin{aligned} &\frac{1}{2} i^\alpha \eta \left[(a_{\alpha\lambda}^\lambda - a_{\alpha 1}^1)\mathbf{1} + (a_{\alpha\lambda}^\lambda + a_{\alpha 1}^1)\mathbf{k} \right], \\ &\frac{1}{2} i^\alpha \eta \left[(a_{\alpha\lambda}^\lambda + a_{\alpha 1}^1)\mathbf{1} - (a_{\alpha\lambda}^\lambda - a_{\alpha 1}^1)\mathbf{k} \right], \end{aligned} \tag{18}$$

where $\eta = 1$ or -1.

(ii) *If*

$$a_{\alpha\lambda}^\lambda = 0 \text{ and } a_{\alpha\lambda-1}^{\lambda-1} = a_{\alpha 2}^2 = a_{\alpha 1}^1 = 0, \text{ where } \lambda = 2^{p+q-2}, \tag{19}$$

the sequences (3) *are constant-valued, starting from some term; it amounts at*

$$-\frac{1}{2} i^\alpha \eta \left[a_{\alpha 1}^1 \mathbf{1} + a_{\alpha 1}^1 \mathbf{k} \right], \quad \textit{where} \quad \eta = 1 \quad \textit{or} \quad -1, \tag{20}$$

(iii) *If* (19) *holds, the sequences* (4) *are periodic of period* 2*, starting from some term. The periods are:*

$$
\begin{aligned}
&\frac{1}{2}i^{\alpha}\eta\left[\left(a_{\alpha\lambda}^{\lambda-1}+a_{\alpha,\lambda-1}^{\lambda}\right)\mathbf{i}+\left(a_{\alpha\lambda}^{\lambda-1}-a_{\alpha,\lambda-1}^{\lambda}\right)\mathbf{j}\right],\\
-&\frac{1}{2}i^{\alpha}\eta\left[\left(a_{\alpha\lambda}^{\lambda-1}+a_{\alpha,\lambda-1}^{\lambda}\right)\mathbf{i}-\left(a_{\alpha\lambda}^{\lambda-1}-a_{\alpha,\lambda-1}^{\lambda}\right)\mathbf{j}\right],
\end{aligned}
\tag{21}
$$

where $\eta=1$ *or* $\eta=-1$.

(iv) *If* (19) *holds, the sequences* (5) *are constant-valued, starting from some term; it amounts at*

$$
-\frac{1}{2}i^{\alpha}\eta\left[\left(a_{\alpha2}^{1}+a_{\alpha1}^{2}\right)\mathbf{i}-\left(a_{\alpha2}^{1}-a_{\alpha1}^{2}\right)\mathbf{j}\right],\quad \textit{where}\quad \eta=1\ \text{ or }\ -1.
\tag{22}
$$

(v) *The sequences* (7) *and* (9) *are periodic of period* 2*, starting from some term. The periods are:*

$$
\left(\frac{1}{2}\mathbf{1}+\frac{1}{2}\mathbf{k},\ \frac{1}{2}\mathbf{1}+\frac{1}{2}\mathbf{k}\right),\ \left(-\frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k},\ -\frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k}\right)
\tag{23}
$$

or

$$
\left(-\frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k},\ -\frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k}\right),\ \left(-\frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k},\ -\frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k}\right)
\tag{24}
$$

in the case of (7)*, and*

$$
\left(-\frac{1}{2i}\mathbf{1}+\frac{1}{2}i\mathbf{k},\ \frac{1}{2i}\mathbf{1}-\frac{1}{2}i\mathbf{k}\right),\ \left(-\frac{1}{2i}\mathbf{1}+\frac{1}{2}i\mathbf{k},\ \frac{1}{2i}\mathbf{1}-\frac{1}{2}i\mathbf{k}\right)
\tag{25}
$$

or

$$
\left(-\frac{1}{2i}\mathbf{1}+\frac{1}{2}i\mathbf{k},\ \frac{1}{2i}\mathbf{1}-\frac{1}{2}i\mathbf{k}\right),\ \left(\frac{1}{2i}\mathbf{1}-\frac{1}{2}i\mathbf{k},\ -\frac{1}{2i}\mathbf{1}+\frac{1}{2}i\mathbf{k}\right)
\tag{26}
$$

in the case of (9).

(vi) *The sequences* (8) *and* (10) *are constant-valued, starting from some term; it amounts at*

$$
\left(-\frac{1}{2}\mathbf{1}+\frac{1}{2}\mathbf{k},\ -\frac{1}{2}\mathbf{1}+\frac{1}{2}\mathbf{k}\right)\ \text{ or }\ \left(\frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k},\ \frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k}\right)
\tag{27}
$$

in the case of (8)*, and*

$$
\left(-\frac{1}{2i}\mathbf{1}-\frac{1}{2}i\mathbf{k},\ \frac{1}{2i}\mathbf{1}+\frac{1}{2}i\mathbf{k}\right)\ \text{ or }\ \left(\frac{1}{2i}\mathbf{1}+\frac{1}{2}i\mathbf{k},\ -\frac{1}{2i}\mathbf{1}-\frac{1}{2}i\mathbf{k}\right)
\tag{28}
$$

in the case of (10).

(vii) *The sequences* (13) *and* (14) *are periodic of period* 2*, starting from some term. The periods are:*

$$
\left(\frac{1}{2i}\mathbf{1}-\frac{1}{2}i\mathbf{k},\ \frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k}\right),\ \left(-\frac{1}{2i}\mathbf{1}+\frac{1}{2}i\mathbf{k},\ \frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k}\right)
\tag{29}
$$

or

$$\left(-\frac{1}{2i}\mathbf{1}+\frac{1}{2}i\mathbf{k}\,,\ \frac{1}{2}\mathbf{1}+\frac{1}{2}\mathbf{k}\right),\quad \left(\frac{1}{2i}\mathbf{1}-\frac{1}{2}i\mathbf{k}\,,\ \frac{1}{2}\mathbf{1}+\frac{1}{2}\mathbf{k}\right) \tag{30}$$

in the case of (13)*, and* (30) *or* (29) *in the case of* (14) *(given* $z \in L^0_\infty$*, the choices* (27) *and* (28) *are mutually correlated).*

(viii) *The sequences* (11) *and* (12) *are periodic of period* 2*, starting from some term. The periods are:*

$$\left(\frac{1}{2}\mathbf{1}+\frac{1}{2}\mathbf{k}\,,\ -\frac{1}{2}\mathbf{1}+\frac{1}{2}\mathbf{k}\right),\quad \left(-\frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k}\,,\ -\frac{1}{2}\mathbf{1}+\frac{1}{2}\mathbf{k}\right) \tag{31}$$

or

$$\left(-\frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k}\,,\ \frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k}\right),\quad \left(\frac{1}{2}\mathbf{1}+\frac{1}{2}\mathbf{k}\,,\ \frac{1}{2}\mathbf{1}-\frac{1}{2}\mathbf{k}\right). \tag{32}$$

The proof is analogous to that of the corresponding theorem given in [4–6]. However, in order to understand two important differences:

$i\mathbf{j}, \mathbf{i}, i\mathbf{k}$ instead of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and $i^\alpha A^q_\alpha$ instead of A^q_α, $\alpha = 1, 2, \ldots, 2p+2q-3$,

we give a proof in the case of α odd and $\alpha > 2p$.

2. Proof of the Periodicity Theorem for α odd and $\alpha > 2p$. In order to calculate

$$\begin{aligned}
&\left(g^\alpha_q\left(z^-_-\left(p+q-1-\left[\frac{1}{2}\alpha\right]\right)\right),\ g^\alpha_q\left(z^+_-\left(p+q-1\right)-\left[\frac{1}{2}\alpha\right]\right)\right)\\
&\quad=\left(g^\alpha_q\left(\frac{1}{2\sqrt{2}}\frac{m}{2^n}(1-i)-\frac{1}{2\sqrt{2}}\frac{1}{2^{p+q-1-[1/2\alpha]}}\right),\right.\\
&\qquad\left. g^\alpha_q\left(\frac{1}{2\sqrt{2}}\frac{m}{2^n}(1-i)-\frac{i}{2\sqrt{2}}\frac{1}{2^{p+q-1-[1/2\alpha]}}\right)\right),
\end{aligned} \tag{33}$$

$m = 0, \pm 1, \ldots, \pm(2^n-1)$; $n = 0, 1, \ldots$, we have to look for

$$\left(g^\alpha_{q-s}\left(z^-_-\left(p+q-1-\left[\frac{1}{2}\alpha\right]\right)\right),\ g^\alpha_{q-s}\left(z^+_-\left(p+q-1-\left[\frac{1}{2}\alpha\right]\right)\right)\right), \tag{34}$$

$s = 1, 2, \ldots$. To this end in the case of the Clifford-type fractal $\sum_5 = (Q^5_q)$, $p = 2[1, 2, 11]$ we consider the table (cf. [4], Fig. 1), where rows numbered with positive integers, corresponding to *odd* numbers, represent

$$\begin{aligned}
&\left\{z^-_-\left(p+q-1-\left[\frac{1}{2}\alpha\right]\right)\right\}\cup\left\{z^+_-\left(p+q-1-\left[\frac{1}{2}\alpha\right]\right)\right\}\\
&\quad=\left\{z^-_-\left(p+q-3\right)\right\}\cup\left\{z^+_-\left(p+q-3\right)\right\}
\end{aligned} \tag{35}$$

on the q-th iteration step and columns represent the configuration related to $s_m = m/2^n$; hereafter, for $z \in L_\infty$, we are also using the notation

$$s = \frac{1}{\sqrt{2}}\left((\operatorname{re} z - \operatorname{im} z) + \frac{1}{2}\right).$$

In principle, the configuration (35) coincides for $p = 2$ with that of Sec. 6 in [4], but we have to take into account that the number of basic squares was $4^{p+q-2} = 4^q$, and now it amounts at $4^{p+q-1} = 4^{q+1}$.

The complete configuration related to (35) consists of four values of the generating function if z belongs to the interior of the basic square $Q^{\alpha j}_{qk}$ corresponding to 16 values if z is a vertex (of course, it may happen that some of these values coincide). In our case we have the following possibilities for the four values corresponding to the direction perpendicular to L_∞ determined by the points $z^-_-(q-1)$ and $z^+_-(q-1)$ starting from some term of (9):

$$\begin{array}{cc} \boxed{i} & \boxed{-i} \\ \boxed{0} & \boxed{0} \end{array} \quad \text{or} \quad \begin{array}{cc} \boxed{-i} & \boxed{i} \\ \boxed{0} & \boxed{0} \end{array}.$$

As far as the direction of L_∞ corresponding to $z \in L_\infty$ is concerned, we have only one possibility: $\boxed{i}\ \boxed{-i}$ or $\boxed{-i}\ \boxed{i}$. The complete configuration becomes:

$$\boxed{}\begin{array}{c} \boxed{i} \\ \boxed{0} \end{array}\boxed{}\boxed{}\begin{array}{c} \boxed{-i} \\ \boxed{0} \end{array} \quad \text{or} \quad \boxed{}\begin{array}{c} \boxed{-i} \\ \boxed{0} \end{array}\boxed{}\boxed{}\begin{array}{c} \boxed{i} \\ \boxed{0} \end{array}.$$

For each n the first period is indicated with help of the upper parts of two bigger squares: each upper part containing four small squares $\boxed{i}$ and four small rectangles $\boxed{}$.

Since the direct proof of the statement (v) of the Periodicity Theorem seems to be more complicated that the preceding ones, it is instructive to illustrate the determining procedure (34) for the complete structure (33) of the gradating function with respect to $(z^-_-, z^+_-, z^-_+, z^+_+)$ in the case of $\sum_5$ and $p = 2$ by decomposing the lifting inverse to the projection (34) with respect to s, into three subliftings, covering the range of Fig. 1 in [4], as indicated on Fig. 2 in [4]:

$$\begin{aligned} &(q = 3) \xrightarrow{s=11} (4) \xrightarrow{10} \dots \xrightarrow{7} (8), \\ &(q = 8) \xrightarrow{s=6} (9) \xrightarrow{5} (10) \xrightarrow{4} (11), \\ &(q = 11) \xrightarrow{s=3} (12) \xrightarrow{2} (13) \xrightarrow{1} (14). \end{aligned}$$

The data for the initial lifting $s = 11$ corresponding to $q = 3$ can easily be checked out from Fig. 3 in [4], the square corresponding to

$$p = 2, \quad \alpha = 4, 5; \quad q = 3, \quad h = 2.$$

In the case of $\sum_7 = (Q^7_q)$, $p = 3$ thanks to starting with q always from 1, the corresponding table differs from the previous one by minor changes only, caused by the fact that the number of basic squares has increased, for $q = 1$ from 16 to 64; cf. Fig. in [4], the squares corresponding to

$$\begin{aligned} &p = 2, \quad \alpha = 4, 5; \quad q = 1, \quad h = 0, \\ &p = 2, \quad h = 6, 7; \quad q = 1, \quad h = 0. \end{aligned}$$

Consequently, within the range of Fig. 1 in [4], the changes concern $q \leq 9$ only; they are shown in Fig. 4 in [4].

Then we can proceed by induction with respect to p, considering $\sum_\alpha$, $\alpha = 2p+1$. We observe that, by the definition of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$, the matrices expressing the periods can be given in the form (25) and (26), as desired.

3. The second and third Periodicity Theorem in para-quaternionic formulation. The following generalizations of Periodicity Theorems (iii) and (iv) are obtained by analogy with those established in [4].

If we remove the second condition from (19):

$$a^{\lambda}_{\alpha\lambda} = 0 \text{ for } \lambda = 2^{p+q-2} \tag{36}$$

then the periods (21) have a more general form

$$i^{\alpha} \begin{pmatrix} \eta a^{\lambda-1}_{\alpha,\lambda-1} & \eta a^{\lambda}_{\alpha,\lambda-1} \\ \eta a^{\lambda-1}_{\alpha\lambda} & 0 \end{pmatrix}, \quad i^{\alpha} \begin{pmatrix} -\eta a^{\lambda-1}_{\alpha,\lambda-1} & -\eta a^{\lambda}_{\alpha,\lambda-1} \\ -\eta a^{\lambda-1}_{\alpha\lambda} & 0 \end{pmatrix}, \tag{37}$$

where $\eta = 1$ or -1, and the para-quaternionic form, instead of (21) reads:

$$\begin{aligned} &\tfrac{1}{2} i^{\alpha}\eta \left[a^{\lambda-1}_{\alpha,\lambda-1}\mathbf{1} + (a^{\lambda-1}_{\alpha\lambda} + a^{\lambda}_{\alpha,\lambda-1})\mathbf{i} + (a^{\lambda-1}_{\alpha\lambda} - a^{\lambda}_{\alpha,\lambda-1})\mathbf{j} + a^{\lambda-1}_{\alpha,\lambda-1}\mathbf{k}\right], \\ -&\tfrac{1}{2} i^{\alpha}\eta \left[a^{\lambda-1}_{\alpha,\lambda-1}\mathbf{1} - (a^{\lambda-1}_{\alpha\lambda} + a^{\lambda}_{\alpha,\lambda-1})\mathbf{i} - (a^{\lambda-1}_{\alpha\lambda} - a^{\lambda}_{\alpha,\lambda-1})\mathbf{j} - a^{\lambda-1}_{\alpha,\lambda-1}\mathbf{k}\right]. \end{aligned} \tag{38}$$

Therefore we have arrived at the following generalization of Periodicity Theorem (iii):

Second Periodicity Theorem (para-quaternionic formulation). (i) *If* (36) *holds, the sequences* (4) *are periodic of period* 2*, starting from some term. The periods are* (38)*, where* $\eta = 1$ *or* -1.

(ii) *The periodicity of the sequences* (4) *always starts from their* n*-th term or earlier whenever* m *in the definition of* $z \in L^0_\infty$ *is odd. If* m *is even, of the form* $m = \mu \cdot 2^\nu$*, where* ν *is odd, the periodicity of* (4) *starts from the* $n-\nu$*-th term or earlier. In particular, such a situation appears in the cases of Pauli matrices* σ_1 *and* σ_2*, where the periodicity, for* m *odd, starts exactly from the* n*-th term. For* m *even the periodicity starts exactly from the* $n-\nu$*-th term.*

If we replace (19) by a less restrictive condition (36), then the one-element periods (22) have a more general form

$$i^{\alpha} \begin{pmatrix} -\eta a^{1}_{\alpha 1} & -\eta a^{2}_{\alpha 1} \\ -\eta a^{1}_{\alpha 2} & -\eta a^{2}_{\alpha 2} \end{pmatrix}, \tag{39}$$

where $\eta = 1$ or -1, and the para-quaternionic form, instead of (22), reads:

$$-\frac{1}{2} i^{\alpha}\eta \left[(a^{1}_{\alpha 1} + a^{2}_{\alpha 2})\mathbf{1} - (a^{1}_{\alpha 2} + a^{2}_{\alpha 1})\mathbf{i} - (a^{1}_{\alpha 2} - a^{2}_{\alpha 1}) - (a^{1}_{\alpha 1} - a^{2}_{\alpha 2})\mathbf{k}\right]. \tag{40}$$

Therefore we have arrived at the following generalization of Periodicity Theorem (iv):

Third Periodicity Theorem. (i) *If* (36) *holds, the sequences* (5) *are constant-valued, starting from some term. It amounts at* (40), *where* $\eta = 1$ *or* -1.

(ii) *The constancy of the sequences* (5) *always starts from their* n*-th term or earlier, whenever* m *in the definition of* $z \in L_\infty^0$ *is odd. If* m *is even, of the form* $m = \mu \cdot 2^\nu$, *where* ν *is odd, the constancy of* (5) *starts from the* $n - \nu$*-th term or earlier. In particular, such a situation appears in the cases of Pauli matrices* σ_1 *and* σ_2, *where the constancy, for* m *odd, starts exactly from the* n*-th term. For* m *even the periodicity starts exactly from the* $n - \nu$*-th term.*

4. Perspectives of combining structure fractals with para-quaternionic geometry. The periodicity theorems for structure fractals in para--quaternionic formulation give wide research perspectives which we plan to explore in the future:

1. Physical interpretation of a generic subspace of a para-quaternionic Hermitian vector space [3, 14].

2. Algebraical aspects related with the five-dimensional space-time [12, 15].

3. Noncommutativity and phase-space theorem referring to the well--known Gelfand–Naĭmark theorem (1943): Let $\mathcal{A}$ be a commutative C^*-algebra and M denote the set of maximal ideals of $\mathcal{A}$. Then, equipped with a natural topology, M is a locally compact topological space, and $\mathcal{A} = C_0(M)$, where C_0 denotes the C^*-algebra of continuous functions of M vanishing at infinity [5, 16].

4. Finsler geometry of holomorphical and projectivized bundles, in particular, extension of the quaternionic Randers models to a para-quaternionic geometry [8].

5. Supercomplex structures in complex Finslerian quantum mechanics [10].

6. Para-quaternionic geometry vs. openness and dissipativity of the system [9].

7. Torsion-depending deformations within the electromagnetic spaces [6].

8. Complex Randersian physics vs. isospectral deformations [7].

9. Complex gauge connections of interacting fields [7].

10. Spin connections of the triple of correlations diffeomorphism ξ of two physical systems, ξ-morphism e of related vector bundles, and the metric F_0 [7].

11. Generalized Dirac–Maxwell systems [13, 16].

12. Complex-analytical approach to Dirac–Maxwell systems vs. physical demands.

13. The solenoidal and nanosolenoidal parts of the generalized Yang–Mills equations as observed on the canonical principal fibre bundle.

14. Simplifying the external field in terms of the metric and connection.

15. General mathematical and physical conclusions of combining structure fractals with para-quaternionic geometry.

Acknowledgments. Research of the first named author was partially supported by the Ministry of Science and Higher Education grant PB1 P03A 00126 (Sections 1–2 of the paper), and partially by the grant of the University of Łódź no. 505/692 (Sections 3–4). Research of the second named author was carried out under the programme of GNSAGA-INDAM of C.N.R. and PRIN07 "Riemannian metrics and differentiable structures" of MIUR (Italy) and within the framework of the direct cooperation agreement between the Universities of Roma "La Sapienza" and Łódź.

References

[1] Kigami, J., *Analysis on Fractals*, Cambridge Tracts in Mathematics 143, Cambridge University Press, Cambridge, 2001.

[2] Kovacheva, R., Ławrynowicz, J. and Nowak-Kępczyk, M., *Critical dimension* 13 *in approximation related to fractals of algebraic structure*, XIII Conference on Analytic Functions, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. **37** (2002), 77–102.

[3] Ławrynowicz, J., Marchiafava, S. and Nowak-Kępczyk, M., *Quaternionic background of the periodicity of petal and sepal structures in some fractals of the flower type*, Proceedings of the 5th ISAAC Congress, Catania, July 25–30, 2005: More Progresses in Analysis, ed. by H. Begehr and F. Nicolosi, World Scientific, Singapore, 2009, pp. 987–996.

[4] Ławrynowicz, J., Marchiafava, S. and Nowak-Kępczyk, M., *Periodicity theorem for structure fractals in quaternionic formulation*, Int. J. Geom. Methods Mod. Phys. **5** (2006), 1167–1197.

[5] Ławrynowicz, J., Marchiafava, S. and Nowak-Kępczyk, M., *Cluster sets and periodicity in some structure fractals*, Topics in Contemporary Differential Geometry, Complex Analysis and Mathematical Physics, ed. by S. Dimiev and K. Sekigawa, World Scientific, New Jersey–Singapore–London, 2007, pp. 179–195.

[6] Ławrynowicz, J., Marchiafava, S. and Nowak-Kępczyk, M., *Mathematical outlook of fractals and related to simple orthorhombic Ising–Onsager–Zhan lattices*, Complex Strucures, Intergrability, Vector Fields, and Mathematical Physics, ed. by S. Dimiev and K. Sekigawa, World Scientific, Singapore, 2009, pp. 156–166.

[7] Ławrynowicz, J., Marchiafava, S. and Nowak-Kępczyk, M., *Applied peridicity theorem for structure fractals in quaternionic formulation*, Applied Complex and Quaternionic Approximation, ed. by R. K,Kovacheva, J. Ławrynowicz and S. Marchiafava, Ediz. Nuova Cultura Univ. "La Sapienza", Roma, 2009, pp. 93–122.

[8] Ławrynowicz, J., Nôno, K. and Suzuki, O., *Hyperfunctions on fractal boundaries: meromorphic Schauder basis for a fractal set*, Bull. Soc. Sci. Lett. Łódź Sér. Déform. **53** (2007), 63–74.

[9] Ławrynowicz, J., Nowak-Kępczyk, M. and Suzuki, O., *A duality theorem for inoculated graded fractal bundles vs. Cuntz algebras and their central extensions*, Int. J. Pure Appl. Math. **52** (2009), 315–338.

[10] Ławrynowicz, J., Ogata, T. and Suzuki, O., *Differential and integral calculus for Schauder basis on a fractal set* (I) *(Schauder basis* 80 *years after)*, Lvov Mathematical School in the Period 1915–45 as Seen Today, ed. by B. Bojarski, J. Ławrynowicz and Ya. G. Prytuła, Banach Center Publications 87, Institute of Mathematics, Polish Academy of Sciences, Warszawa, 2009, pp. 115–140.

[11] Ławrynowicz, J., Suzuki, O., *Periodicity theorem for graded fractal bundles related to Clifford structures*, Int. J. Pure Appl. Math. **24** (2005), no. 2, 181–209.

[12] Ławrynowicz, J., Suzuki, O. and Castillo Alvarado, F. L., *Basic properties and applications of graded fractal bundles related to Clifford structures: An introduction*, Ukrain. Math. Zh. **60** (2008), 603–618.

[13] Lounesto, P., *Clifford Algebra and Spinors*, London Math. Soc. Lecture Notes 239, Cambridge Univ. Press, Cambridge, 1997, 2nd ed. (vol. 286), ibid. 2001.

[14] Vaccaro, M., *Subspaces of a para-quaternionic Hermitian vector space*, 2010, ArXiv:1011.2947v1 [math.D6]; Internat. J. of Geom. Methods in Modern Phys. **8** (2011), in print.

[15] Vaccaro, M., *Basics of linear para-quaternionic geometry I. Hermitian para-type structure on a real vector space*, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. **61** (2011), no. 1, 23–36.

[16] Vaccaro, M., *Basics of linear para-quaternionic geometry II. Decomposition of a generic subspace of a para-quaternionic hermitian vector space*, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. **61** (2011), no. 2, 17–34.

Julian Ławrynowicz
Institute of Physics
University of Łódź
Pomorska 149/153
PL-90-236 Łódź
Poland

Institute of Mathematics
Polish Academy of Sciences
Łódź Branch, Banacha 22
PL-90-38 Łódź
Poland
e-mail: `jlawryno@uni.lodz.pl`

Massimo Vaccaro
Dipartimento dell'Ingegneria di Informazione
e Matematica Applicata
Università di Salerno
I-84084 Fisciano (SA)
Italy
e-mail: `masimo_vaccaro@liberto.it`

Received July 20, 2011