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On a modification of the Poisson integral operator

In memory of Professor Jan Krzyż

Abstract. Given a quasisymmetric automorphism γ of the unit circle $\mathbb{T}$ we define and study a modification P_γ of the classical Poisson integral operator in the case of the unit disk $\mathbb{D}$. The modification is done by means of the generalized Fourier coefficients of γ. For a Lebesgue's integrable complex-valued function f on $\mathbb{T}$, $\mathrm{P}_\gamma[f]$ is a complex-valued harmonic function in $\mathbb{D}$ and it coincides with the classical Poisson integral of f provided γ is the identity mapping on $\mathbb{T}$. Our considerations are motivated by the problem of spectral values and eigenvalues of a Jordan curve. As an application we establish a relationship between the operator P_γ, the maximal dilatation of a regular quasiconformal Teichmüller extension of γ to $\mathbb{D}$ and the smallest positive eigenvalue of γ.

Introduction. A number of important problems in the potential theory of the complex plane $\mathbb{C}$ can be reduced to a linear integral equation of Fredholm type with the Neumann–Poincaré kernel k or its transposition. This kernel is assigned to a rectifiable and sufficiently smooth Jordan curve $\Gamma \subset \mathbb{C}$ by the formula

$$k(\zeta, z) := -\frac{1}{\pi}\frac{\partial}{\partial \vec{n}_\zeta} \log|\zeta - z|, \quad \zeta, z \in \Gamma,\ \zeta \neq z, \tag{0.1}$$

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where $\dfrac{\partial}{\partial \vec{n}_\zeta}$ denotes the derivative along the interior normal of Γ at a point ζ. For details the reader is referred to e.g. [3], [5], [32]. A very short but essential survey of basic problems can be found in [30]. Let us recall that a real number λ is called a *Fredholm eigenvalue of* Γ if it is an eigenvalue of the kernel k, i.e. if there exists a real-valued function μ integrable on Γ and non-constant almost everywhere (a.e. for brevity), which satisfies the homogeneous integral equation

$$(0.2) \qquad \mu(z) = \lambda \int_\Gamma k(\zeta, z)\mu(\zeta)|d\zeta|\,, \quad \text{for almost every } z \in \Gamma.$$

The theory of Fredholm eigenvalues of a Jordan curve has been intensively studied by a number of eminent mathematicians like Ahlfors, Bergman and Schiffer, next by Schober and Springer, and lately by Krushkal, Krzyż and Kühnau.

Two Krzyż's ideas seem to be especially important in the contemporary theory of eigenvalues of a Jordan curve. First of all he observed in [11] and [10] that every Fredholm eigenvalue λ of a sufficiently regular Jordan curve Γ can be expressed equivalently by a pair of continuous functions $F : \mathrm{cl}(\Omega) \to \mathbb{C}$ and $F_* : \mathrm{cl}(\Omega_*) \to \mathbb{C}$ which are analytic in the domains Ω and $\Omega_* \ni \infty$ complementary to Γ, and satisfy the following boundary assumptions on Γ:

$$(0.3) \qquad \operatorname{Im} F = \operatorname{Im} F_* \quad \text{and} \quad (1-\lambda)\operatorname{Re} F = (1+\lambda)\operatorname{Re} F_*\,.$$

The notation $\mathrm{cl}(A)$ means the closure of a set $A \subset \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ in the spherical metric. It is worth noting here that all Fredholm eigenvalues of Γ can be represented without using the Neumann–Poincaré kernel. This idea appears implicitly also in the works of Kühnau; cf. e.g. [17]. Having disposed of the equalities (0.3), Krzyż proposed to modify them by using the continuous mappings $H : \mathrm{cl}(\mathbb{D}) \to \mathrm{cl}(\Omega)$ and $H_* : \mathrm{cl}(\mathbb{D}_*) \to \mathrm{cl}(\Omega_*)$ which are conformal in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and in $\mathbb{D}_* := \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, respectively. Then the mappings $G := F \circ H$ and $G_* := F_* \circ H_*$ are continuous and analytic in $\mathbb{D}$ and $\mathbb{D}_*$, respectively. Moreover, in view of (0.3) they satisfy on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ the following equalities

$$(0.4) \qquad \operatorname{Im} G = \operatorname{Im} G_* \circ \gamma \quad \text{and} \quad (1-\lambda)\operatorname{Re} G = (1+\lambda)\operatorname{Re} G_* \circ \gamma\,,$$

where $\gamma := H_*^{-1} \circ H : \mathbb{T} \to \mathbb{T}$ is so-called *welding homeomorphism of* Γ. This way the eigenvalue problem for a Jordan curve Γ can be reduced to a new problem of studying G, G_* and λ satisfying the equalities (0.4) for a given homeomorphic self-mapping γ of the unit circle. The author, encouraged by Krzyż, pursued this line of research in several works by introducing and studying so-called eigenvalues of an automorphism of the unit circle;

cf. e.g. [14], [24], [21], [22], [23] and [25]. Therefore the author is much indebted to professor Jan Krzyż for introducing him to the theory of Fredholm eigenvalues of a Jordan curve.

The definition of an eigenvalue of a quasisymmetric automorphism γ of the unit circle is recalled in Section 3. This is done by applying the harmonic conjugation operator and certain operator $\boldsymbol{B}_\gamma$ assigned to γ. Then the smallest positive eigenvalue of γ is described by the Poisson integral modified by γ; cf. Theorem 3.1. This modification P_γ is defined by means of the generalized Fourier coefficients of γ in Section 1. Then the basic properties of the operator P_γ are studied. In Section 2 we recall the definition of the operator $\boldsymbol{B}_\gamma$ and prove an important relationship between the operators P_γ and $\boldsymbol{B}_\gamma$; cf. Theorem 2.1. Then we characterize the norm of $\boldsymbol{B}_\gamma(\boldsymbol{f})$ by means of the Dirichlet integral $\mathrm{D}[\mathrm{P}_\gamma[f]]$ (Corollary 2.2), where $\boldsymbol{f}$ is the abstract class of a real-valued function $f \in \mathrm{H}^{1/2}$ with respect to the equivalence relation $\doteqdot$. The class $\mathrm{H}^{1/2}$ consists of all Lebesgue's integrable complex-valued functions on $\mathbb{T}$ whose harmonic extensions to $\mathbb{D}$ have finite Dirichlet's integral. The relation $\doteqdot$ identifies any complex-valued and Lebesque's measureable functions on $\mathbb{T}$ whose difference is a constant function a.e. in $\mathbb{T}$.

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1. The Poisson integral modified by a quasisymmetric automorphism of the unit circle. Let us denote by $\mathrm{Hom}^+(\mathbb{T})$ the class of all sense-preserving homeomorphic self-mappings of $\mathbb{T}$. For $K \geq 1$ let $\mathrm{Q}(\mathbb{T}, K)$ be the class of all $\gamma \in \mathrm{Hom}^+(\mathbb{T})$ which admit a K-quasiconformal extension to $\mathbb{D}$. Homeomorphisms belonging to the class $\mathrm{Q}(\mathbb{T}) := \bigcup_{K\geq 1} \mathrm{Q}(\mathbb{T}, K)$ were called by Krzyż as *quasisymmetric automorphisms of the unit circle*; cf. [12] and [13]. He noticed that each $f \in \mathrm{Q}(\mathbb{T})$ can be described by a similar condition to the well-known Beurling–Ahlfors quasisymmetricity condition; cf. [1]. For other characterizations of the class $\mathrm{Q}(\mathbb{T})$ see [33] and [27].

Let $\mathrm{L}^0(\mathbb{T})$ stand for the class of all Lebesgue's measurable functions $f : \mathbb{T} \to \mathbb{C}$. We denote by $\mathrm{L}^1(\mathbb{T})$ the class of all $f \in \mathrm{L}^0(\mathbb{T})$ which are integrable on $\mathbb{T}$ with respect to the Lebesgue arc-length measure, i.e. $\int_{\mathbb{T}} |f(z)||\mathrm{d}z| < +\infty$. Let $\mathrm{P}[f]$ be the Poisson integral of a function $f \in \mathrm{L}^1(\mathbb{T})$, i.e.

$$
\begin{aligned}
\mathrm{P}[f](z) &:= \frac{1}{2\pi}\int_{\mathbb{T}} f(u)\,\mathrm{Re}\,\frac{u+z}{u-z}|\mathrm{d}u| \\
&= \hat{f}(0) + \sum_{n=1}^{\infty}\hat{f}(n)z^n + \sum_{n=1}^{\infty}\hat{f}(-n)\bar{z}^n\,, \quad z \in \mathbb{D},
\end{aligned} \tag{1.1}
$$

where

$$
\hat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{T}} f(u)u^{-n}|\mathrm{d}u|\,, \quad n \in \mathbb{Z}. \tag{1.2}
$$

It is well known that $\mathrm{P}[f]$ is a complex-valued harmonic function on $\mathbb{D}$. Moreover, if the function f is continuous, then the function $\mathrm{P}[f]$ is the unique solution to the Dirichlet problem for the boundary function f, which means that for every $z \in \mathbb{T}$, $\mathrm{P}[f](\zeta) \to f(z)$ as $\mathbb{D} \ni \zeta \to z$.

Given a function $f \in \mathrm{L}^0$ and $m, n \in \mathbb{Z}$ we define

$$\hat{f}(m,n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}t})^m \mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{T}} f(u)^m u^{-n}|\mathrm{d}u|\,, \tag{1.3}$$

provided the respective functions are integrable on $\mathbb{T}$. If $m = 1$, then (1.3) takes the form of

$$\hat{f}(1,n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t = \hat{f}(n)\,, \tag{1.4}$$

and so $\hat{f}(1,n)$ is just the nth Fourier coefficient of the function f. This justifies to call $\hat{f}(m,n)$ the (m,n)*-generalized Fourier coefficient of the function* f. If f satisfies the following condition

$$0 < \operatorname*{ess\,inf}_{z\in\mathbb{T}} |f(z)| \le \operatorname*{ess\,sup}_{z\in\mathbb{T}} |f(z)| < +\infty\,,$$

then $\hat{f}(m,n)$ is well defined for any $m, n \in \mathbb{Z}$. In particular, if $f \in \mathrm{Hom}^+(\mathbb{T})$, then all generalized Fourier coefficients $\hat{f}(m,n)$, $m, n \in \mathbb{Z}$, are well defined. In [26] the following result was proved.

Theorem A. *Given $K \ge 1$ let $\gamma \in \mathrm{Q}(\mathbb{T}, K)$. If $\mathbb{Z} \ni n \mapsto \lambda_n \in \mathbb{C}$ is a sequence such that*

$$\sum_{n=-\infty}^{\infty} |n||\lambda_n|^2 < +\infty\,, \tag{1.5}$$

then for each $n \in \mathbb{Z}$ the sequence $\mathbb{N} \ni p \mapsto \sum_{m=-p}^{p} \sqrt{|n|}\hat{\gamma}(m,n)\lambda_m$ is convergent as $p \to \infty$ and

$$\frac{1}{K}\sum_{n=-\infty}^{\infty} |n||\lambda_n|^2 \le \sum_{n=-\infty}^{\infty} \left| \sum_{m=-\infty}^{\infty} \sqrt{|n|}\hat{\gamma}(m,n)\lambda_m \right|^2 \le K \sum_{n=-\infty}^{\infty} |n||\lambda_n|^2. \tag{1.6}$$

Here and subsequently, we write $\sum_{n=-\infty}^{\infty} c_n := \lim_{m\to\infty} \sum_{n=-m}^{m} c_n$ for any sequence $\mathbb{Z} \ni n \mapsto c_n \in \mathbb{C}$, provided the limit exists.

Note that the inequalities (1.6) look similarly to the Grunsky inequalities for holomorphic functions in the classes $\Sigma(k)$, $0 \le k \le 1$; cf. [28, Sect. 3.1 and 9.4]. Due to the works of R. Kühnau [18], [19], [20] and Y. Shen [31] we know that the inequalities (1.6) can be improved.

Let $\mathrm{D}[F]$ denote the Dirichlet integral of a function $F : \mathbb{D} \to \mathbb{C}$, which is a.e. differentiable in $\mathbb{D}$, i.e.

$$\mathrm{D}[F] := \int_{\mathbb{D}} \left(\left|\frac{\partial F}{\partial x}\right|^2 + \left|\frac{\partial F}{\partial y}\right|^2 \right) \mathrm{d}x\mathrm{d}y = 2\int_{\mathbb{D}} (|\partial F|^2 + |\bar{\partial} F|^2)\mathrm{d}x\mathrm{d}y\,, \tag{1.7}$$

where

$$\partial F := \frac{1}{2}\left(\frac{\partial F}{\partial x} - \mathrm{i}\frac{\partial F}{\partial y}\right) \ , \quad \bar{\partial} F := \frac{1}{2}\left(\frac{\partial F}{\partial x} + \mathrm{i}\frac{\partial F}{\partial y}\right)$$

are so-called the formal derivatives of F. If $F : \mathbb{D} \to \mathbb{C}$ is a harmonic mapping in $\mathbb{D}$ given by the series expansion

$$F(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n}\overline{z}^n \,, \quad z \in \mathbb{D},$$

with coefficients $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$, then integrating by substitution we obtain

$$\begin{aligned}
\int_{\mathbb{D}} (|\partial F|^2 + |\bar{\partial} F|^2)\mathrm{d}x\mathrm{d}y &= \lim_{R\to 1^-}\lim_{p\to\infty}\int_0^R\int_0^{2\pi}\Bigg(\Bigg|\sum_{n=1}^{p} n a_n r^{n-1}\mathrm{e}^{\mathrm{i}(n-1)t}\Bigg|^2 \\
&\quad + \Bigg|\sum_{n=1}^{p} n a_{-n} r^{n-1}\mathrm{e}^{-\mathrm{i}(n-1)t}\Bigg|^2\Bigg) r\mathrm{d}t\mathrm{d}r \\
&= \lim_{R\to 1^-}\int_0^R 2\pi\Bigg(\sum_{n=1}^{\infty} n^2|a_n|^2 r^{2n-1} + \sum_{n=1}^{\infty} n^2|a_{-n}|^2 r^{2n-1}\Bigg)\mathrm{d}r \\
&= \pi\lim_{R\to 1^-}\sum_{n=-\infty}^{\infty}|n||a_n|^2 R^{2n} = \pi\sum_{n=-\infty}^{\infty}|n||a_n|^2 \,,
\end{aligned}$$

and consequently,

$$\mathrm{D}[F] = 2\pi\sum_{n=-\infty}^{\infty}|n||a_n|^2 \,; \tag{1.8}$$

cf. [26, (1.2)]. Using Theorem A we can modify the Poisson integral $\mathrm{P}[f]$ as follows. From (1.1) and (1.8) it follows that

$$\mathrm{D}[\mathrm{P}[f]] = 2\pi\sum_{n=-\infty}^{\infty}|n||\hat{f}(n)|^2 \,, \quad f \in \mathrm{L}^1(\mathbb{T}). \tag{1.9}$$

Then for a given $f \in \mathrm{H}^{1/2} := \{h \in \mathrm{L}^1(\mathbb{T}) : \mathrm{D}[\mathrm{P}[h]] < +\infty\}$ we have

$$\sum_{n=-\infty}^{\infty}|n||\hat{f}(n)|^2 = \frac{1}{2\pi}\,\mathrm{D}[\mathrm{P}[f]] < +\infty \,. \tag{1.10}$$

Applying now Theorem A for an arbitrarily fixed $K \geq 1$ and $\gamma \in \mathrm{Q}(\mathbb{T}, K)$ and the sequence $\mathbb{Z} \ni n \mapsto \lambda_n := \hat{f}(n)$ we know that for each $n \in \mathbb{Z}\setminus\{0\}$ the sequence $\mathbb{N} \ni p \mapsto \sum_{m=-p}^{p}\hat{\gamma}(m,n)\lambda_m$ is convergent as $p \to \infty$ and we may define

$$\hat{f}(0;\gamma) := \hat{f}(0) \ \text{ and } \ \hat{f}(n;\gamma) := \lim_{p\to\infty}\sum_{m=-p}^{p}\hat{\gamma}(m,n)\hat{f}(m) \,, \ n \in \mathbb{Z}\setminus\{0\}. \tag{1.11}$$

Moreover, by the second inequality in (1.6),

$$\sum_{n=-\infty}^{\infty} |n||\hat{f}(n;\gamma)|^2 \le K \sum_{n=-\infty}^{\infty} |n||\hat{f}(n)|^2 . \tag{1.12}$$

This means that the operator $\mathrm{H}^{1/2} \ni f \mapsto \mathrm{P}_\gamma[f]$ is well defined by the formula

$$\mathrm{P}_\gamma[f](z) \coloneqq \sum_{n=0}^{\infty} \hat{f}(n;\gamma) z^n + \sum_{n=1}^{\infty} \hat{f}(-n;\gamma)\overline{z}^n, \quad z \in \mathbb{D}, \tag{1.13}$$

and by (1.8),

$$\mathrm{D}[\mathrm{P}_\gamma[f]] = 2\pi \sum_{n=-\infty}^{\infty} |n||\hat{f}(n;\gamma)|^2 , \quad f \in \mathrm{H}^{1/2},\ \gamma \in \mathrm{Q}(\mathbb{T}). \tag{1.14}$$

We call P_γ the *Poisson integral operator modified by* γ. Applying the operator P_γ and the equalities (1.9), (1.11) and (1.14), we can rewrite the inequalities (1.6) in the following shorter form.

Corollary 1.1. *For all* $K \ge 1$, $\gamma \in \mathrm{Q}(\mathbb{T}, K)$ *and* $f \in \mathrm{H}^{1/2}$,

$$\frac{1}{K} \mathrm{D}[\mathrm{P}[f]] \le \mathrm{D}[\mathrm{P}_\gamma[f]] \le K\, \mathrm{D}[\mathrm{P}[f]] . \tag{1.15}$$

Proof. Given $K \ge 1$ and $\gamma \in \mathrm{Q}(\mathbb{T}, K)$ fix $f \in \mathrm{H}^{1/2}$. Then by (1.10) the sequence $\mathbb{Z} \ni n \mapsto \lambda_n \coloneqq \hat{f}(n)$ satisfies the condition (1.5). Theorem A now shows that

$$\frac{1}{K} \sum_{n=-\infty}^{\infty} |n||\lambda_n|^2 \le \sum_{n=-\infty}^{\infty} |n||\hat{f}(n;\gamma)|^2 \le K \sum_{n=-\infty}^{\infty} |n||\lambda_n|^2 .$$

Combining this with (1.9) and (1.14), we obtain the inequalities (1.15), which is the desired conclusion. □

Remark 1.2. By the definition of the operator P_γ we can infer directly its following properties valid for any $\gamma \in \mathrm{Q}(\mathbb{T})$:

- (i) If γ is the identity mapping on $\mathbb{T}$, then $\mathrm{P}_\gamma = \mathrm{P}$, i.e. the mapping $(f, \gamma) \mapsto \mathrm{P}_\gamma[f]$ generalizes the Poisson operator P;
- (ii) $\mathrm{P}_\gamma[\mu f + \nu g] = \mu\, \mathrm{P}_\gamma[f] + \nu\, \mathrm{P}_\gamma[g]$ as $\mu, \nu \in \mathbb{C}$ and $f, g \in \mathrm{H}^{1/2}$, i.e. the operator P_γ is linear;
- (iii) $\mathrm{P}_\gamma[\overline{f}] = \overline{\mathrm{P}_\gamma[f]}$ as $f \in \mathrm{H}^{1/2}$;
- (iv) $\mathrm{P}_\gamma[\operatorname{Re} f] = \operatorname{Re} \mathrm{P}_\gamma[f]$ and $\mathrm{P}_\gamma[\operatorname{Im} f] = \operatorname{Im} \mathrm{P}_\gamma[f]$ as $f \in \mathrm{H}^{1/2}$.

For the proof we apply (1.3), (1.11) and (1.13) to arbitrarily fixed $\mu, \nu \in \mathbb{C}$ and $f, g \in \mathrm{H}^{1/2}$. If γ is the identity mapping on $\mathbb{T}$, then from (1.3) we conclude that

$$\hat{\gamma}(m, n) = \frac{1}{2\pi} \int_{\mathbb{T}} \gamma(z)^m z^{-n} |\mathrm{d}z| = \frac{1}{2\pi} \int_{\mathbb{T}} z^{m-n} |\mathrm{d}z| = 0 , \quad m, n \in \mathbb{Z},\ m \ne n,$$

and $\hat{\gamma}(m,m) = 1$ as $m \in \mathbb{Z}$, hence and by (1.11) that

$$\hat{f}(n;\gamma) = \lim_{p\to\infty} \sum_{m=-p}^{p} \hat{\gamma}(m,n)\hat{f}(m) = \hat{f}(n)\,, \quad m \in \mathbb{Z},$$

and finally by (1.1) and (1.13) that the property (i) holds.

From (1.11) and (1.2) we see that for every $n \in \mathbb{Z}$,

$$\begin{aligned}(\widehat{\mu f + \nu g})(n;\gamma) &= \lim_{p\to\infty} \sum_{m=-p}^{p} \hat{\gamma}(m,n)(\widehat{\mu f + \nu g})(m) \\ &= \lim_{p\to\infty} \sum_{m=-p}^{p} \big(\mu\hat{\gamma}(m,n)\hat{f}(m) + \nu\hat{\gamma}(m,n)\hat{g}(m)\big) \\ &= \mu \lim_{p\to\infty} \sum_{m=-p}^{p} \hat{\gamma}(m,n)\hat{f}(m) + \nu \lim_{p\to\infty} \sum_{m=-p}^{p} \hat{\gamma}(m,n)\hat{g}(m) \\ &= \mu\hat{f}(n;\gamma) + \nu\hat{g}(n;\gamma)\,.\end{aligned}$$

Hence and by (1.13) we deduce the property (ii).

From (1.3) and (1.2) we see that for every $m,n \in \mathbb{Z}$,

$$\overline{\hat{\gamma}(m,n)} = \frac{1}{2\pi}\int_{\mathbb{T}} \overline{\gamma(z)^m z^{-n}}|\mathrm{d}z| = \frac{1}{2\pi}\int_{\mathbb{T}} \gamma(z)^{-m} z^{n}|\mathrm{d}z| = \hat{\gamma}(-m,-n)$$

as well as

$$\widehat{\overline{f}}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} \overline{f(z)} z^{-n}|\mathrm{d}z| = \overline{\frac{1}{2\pi}\int_{\mathbb{T}} f(z) z^{n}|\mathrm{d}z|} = \overline{\hat{f}(-n)}\,.$$

Hence and by (1.11) we have

$$\begin{aligned}\widehat{\overline{f}}(n;\gamma) &= \lim_{p\to\infty} \sum_{m=-p}^{p} \hat{\gamma}(m,n)\widehat{\overline{f}}(m) = \overline{\lim_{p\to\infty} \sum_{m=-p}^{p} \overline{\hat{\gamma}(m,n)}\hat{f}(-m)} \\ &= \overline{\lim_{p\to\infty} \sum_{m=-p}^{p} \hat{\gamma}(-m,-n)\hat{f}(-m)} = \overline{\lim_{p\to\infty} \sum_{m=-p}^{p} \hat{\gamma}(m,-n)\hat{f}(m)} = \overline{\hat{f}(-n;\gamma)}\end{aligned}$$

as $n \in \mathbb{Z}$. Combining this with (1.13) we conclude that for every $z \in \mathbb{D}$,

$$\begin{aligned}\mathrm{P}_\gamma[\overline{f}](z) &= \widehat{\overline{f}}(0;\gamma) + \sum_{n=1}^{\infty} \widehat{\overline{f}}(n;\gamma)z^n + \sum_{n=1}^{\infty} \widehat{\overline{f}}(-n;\gamma)\overline{z}^n \\ &= \overline{\hat{f}(0;\gamma)} + \sum_{n=1}^{\infty} \overline{\hat{f}(-n;\gamma)}z^n + \sum_{n=1}^{\infty} \overline{\hat{f}(n;\gamma)}\overline{z}^n \\ &= \overline{\hat{f}(0;\gamma) + \sum_{n=1}^{\infty} \hat{f}(-n;\gamma)\overline{z}^n + \sum_{n=1}^{\infty} \hat{f}(n;\gamma)z^n} = \overline{\mathrm{P}_\gamma[f](z)}\,,\end{aligned}$$

which yields the property (iii).

Since $\operatorname{Re} f = (f + \overline{f})/2$ and $\operatorname{Im} f = (f - \overline{f})/(2\mathrm{i})$ we infer the property (iv) from the properties (ii) and (iii).

2. Relationships between operators $\mathbf{P}_\gamma$ and $\boldsymbol{B}_\gamma$. In [24] and [23] the operator $\boldsymbol{B}_\gamma$ was assigned to every $\gamma \in \mathrm{Q}(\mathbb{T})$. We recall now its construction. For all $f, g \in \mathrm{L}^0(\mathbb{T})$ the notation $f \doteqdot g$ means that $f - g$ equals a constant function a.e. in $\mathbb{T}$. It is clear that $\doteqdot$ is an equivalence relation in the class $\mathrm{L}^0(\mathbb{T})$. Let $[f/\doteqdot]$ stand for the abstract class of $f \in \mathrm{L}^0(\mathbb{T})$ with respect to $\doteqdot$. Consider the class

$$\boldsymbol{H} \coloneqq \{[f/\doteqdot] : f \in \operatorname{Re} \mathrm{L}^1(\mathbb{T}) \text{ and } \mathrm{D}[\mathrm{P}[f]] < +\infty\}\,. \tag{2.1}$$

Here and subsequently, we set $\operatorname{Re} X \coloneqq \{\operatorname{Re} f : f \in X\}$ for any family X of complex-valued functions. It can be verified in the standard way that $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$ is a real Hilbert space, where

$$\big\|[f/\doteqdot]\big\|_{\boldsymbol{H}} \coloneqq \sqrt{\frac{1}{2}\,\mathrm{D}[\mathrm{P}[f]]}\,, \quad f \in \operatorname{Re} \mathrm{H}^{1/2}; \tag{2.2}$$

cf. [24, Sect. 2.4]. We adopt the usual notation $\mathrm{C}(\mathbb{T})$ for the class of all complex-valued continuous functions on $\mathbb{T}$. From (2.1), (2.2) and (1.9) it follows that the set $\{[f/\doteqdot] : f \in \operatorname{Re} \mathrm{C}(\mathbb{T})\} \cap \boldsymbol{H}$ is dense in $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$. Moreover, it may be concluded from [24, (2.5.1) and Theorems 2.5.3 and 2.4.3] that the inequalities

$$\frac{1}{K}\,\mathrm{D}[\mathrm{P}[f]] \leq \mathrm{D}[\mathrm{P}[f \circ \gamma]] \leq K\,\mathrm{D}[\mathrm{P}[f]]$$

hold for all $K \geq 1$, $f \in \mathrm{C}(\mathbb{T})$ and $\gamma \in \mathrm{Q}(\mathbb{T}, K)$. Then there exists the unique linear continuous operator $\boldsymbol{B}_\gamma : \boldsymbol{H} \to \boldsymbol{H}$ in $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$ satisfying

$$\boldsymbol{B}_\gamma([f/\doteqdot]) = [f \circ \gamma/\doteqdot]\,, \quad f \in \operatorname{Re} \mathrm{C}(\mathbb{T}) \cap \mathrm{H}^{1/2}\,. \tag{2.3}$$

As a matter of fact, $\boldsymbol{B}_\gamma$ is a linear homeomorphism of the space $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$ onto itself; cf. [24, Corollary 2.5.4]. Various properties of spectral values and eigenvalues of a quasisymmetric automorphism γ of the unit circle were obtained by means of the operator $\boldsymbol{B}_\gamma$ and its norm; cf. [24] and [23]. Note that the operator $\boldsymbol{B}_\gamma$ is defined implicitly by the condition (2.3). From the famous Beurling–Ahlfors result [1] we know that a quasisymmetric automorphism γ of $\mathbb{T}$ does not have to be an absolutely continuous function. Moreover, γ can be even purely singular. Therefore, in such a case the composite mapping $f \circ \gamma$ is not Lebesgue's measurable function in general. In consequence, $f \circ \gamma \notin \mathrm{L}^0(\mathbb{T})$ for certain $f \in \mathrm{H}^{1/2}$, and so the family $\operatorname{Re} \mathrm{C}(\mathbb{T}) \cap \mathrm{H}^{1/2}$ cannot be replaced by $\operatorname{Re} \mathrm{H}^{1/2}$ in (2.3). This means that defining the operator $\boldsymbol{B}_\gamma$ directly by composition of functions fails for a singular $\gamma \in \mathrm{Q}(\mathbb{T})$. This problem was overcome in [24, Sect. 2.5], where the following result was stated:

$$\boldsymbol{B}_\gamma([f/\doteqdot]) = [\mathrm{Tr}[\mathrm{P}[f]] \circ \gamma/\doteqdot]\,, \quad f \in \operatorname{Re} \mathrm{H}^{1/2}; \tag{2.4}$$

cf. [24, (2.5.8)]. Here and later on the symbol $\mathrm{Tr}[F]$ denotes the radial limiting valued function of a function $F : \mathbb{D} \to \mathbb{C}$, i.e. for every $z \in \mathbb{T}$,

$$\mathrm{Tr}[F](z) := \lim_{t \to 1^-} F(tz)$$

as the limit exists, while $\mathrm{Tr}[F](z) := 0$ otherwise. It is well known that $\mathrm{Tr}[\mathrm{P}[f]] = f$ a.e. in $\mathbb{T}$ for every $f \in L^1(\mathbb{T})$; cf. [2, Sect. 1.2]. As a matter of fact, the transformation $\mathrm{H}^{1/2} \ni f \mapsto \mathrm{Tr}[P[f]] \circ \gamma$ was used in (2.4). In what follows we shall describe the operator $\boldsymbol{B}_\gamma$ more directly by means of the transformation $\mathrm{H}^{1/2} \ni f \mapsto \mathrm{Tr}[P_\gamma[f]]$. From (1.13), (1.14) and Corollary 1.1 we deduce that

$$\mathrm{Tr}[\mathrm{P}_\gamma[f]] \in \mathrm{H}^{1/2}, \quad f \in \mathrm{H}^{1/2},\ \gamma \in \mathrm{Q}(\mathbb{T}). \tag{2.5}$$

Theorem 2.1. *For every* $\gamma \in \mathrm{Q}(\mathbb{T})$,

$$\boldsymbol{B}_\gamma([f/ \doteqdot]) = [\mathrm{Tr}[\mathrm{P}_\gamma[f]]/ \doteqdot], \quad f \in \mathrm{Re}\,\mathrm{H}^{1/2}. \tag{2.6}$$

Proof. Fix $\gamma \in \mathrm{Q}(\mathbb{T})$ and $f \in \mathrm{Re}\,\mathrm{H}^{1/2}$. Then $\gamma \in \mathrm{Q}(\mathbb{T}, K)$ for certain $K \geq 1$ and the condition (1.10) holds. Since the function f is real-valued, we conclude from (1.2) that

$$\hat{f}(0) \in \mathbb{R} \quad \text{and} \quad \hat{f}(-n) = \overline{\hat{f}(n)}, \quad n \in \mathbb{N}. \tag{2.7}$$

From (1.10) it follows that the sequence $\mathbb{Z} \ni n \mapsto \lambda_n := \hat{f}(n)$ satisfies the condition (1.5). For every $p \in \mathbb{N}$ we define

$$f_p(z) := \hat{f}(0) + \sum_{n=1}^{p} \hat{f}(n) z^n + \sum_{n=1}^{p} \hat{f}(-n) \overline{z}^n, \quad z \in \mathbb{T}. \tag{2.8}$$

By (2.7) each function f_p, $p \in \mathbb{N}$, is real-valued on $\mathbb{T}$. Moreover, by (1.9) we have

$$\mathrm{D}[\mathrm{P}[f_p]] = 2\pi \sum_{n=-p}^{p} |n| |\hat{f}(n)|^2 \leq 2\pi \sum_{n=-\infty}^{\infty} |n| |\hat{f}(n)|^2 = \mathrm{D}[\mathrm{P}[f]] < +\infty, \tag{2.9}$$

and so $f_p \in \mathrm{H}^{1/2}$. Certainly $f_p \in \mathrm{Re}\,\mathrm{C}(\mathbb{T})$ as $p \in \mathbb{N}$. Thus

$$f_p \in \mathrm{Re}\,\mathrm{C}(\mathbb{T}) \cap \mathrm{H}^{1/2}, \quad p \in \mathbb{N}. \tag{2.10}$$

Then setting $\boldsymbol{f}_p := [f_p/ \doteqdot]$ as $p \in \mathbb{N}$, we deduce from (2.3) that

$$\boldsymbol{B}_\gamma(\boldsymbol{f}_p) = [f_p \circ \gamma/ \doteqdot], \quad p \in \mathbb{N}. \tag{2.11}$$

Since

$$\mathrm{P}[f - f_p] = \mathrm{P}[f] - \mathrm{P}[f_p] = \sum_{n=p+1}^{\infty} \hat{f}(n) z^n + \sum_{n=p+1}^{\infty} \hat{f}(-n) \overline{z}^n, \quad z \in \mathbb{D}, \tag{2.12}$$

we conclude from (1.9) that

$$\begin{aligned}\mathrm{D}[\mathrm{P}[f-f_p]] &= 2\pi \sum_{n=p+1}^{\infty} n|\hat{f}(n)|^2 + 2\pi \sum_{n=p+1}^{\infty} n|\hat{f}(-n)|^2 \\ &= 2\pi \sum_{n=-\infty}^{\infty} |n||\hat{f}(n)|^2 - 2\pi \sum_{n=-p}^{p} |n||\hat{f}(n)|^2 \\ &= \mathrm{D}[\mathrm{P}[f]] - \mathrm{D}[\mathrm{P}[f_p]] \le \mathrm{D}[\mathrm{P}[f]] < +\infty\,, \quad p \in \mathbb{N}.\end{aligned}$$

Hence $f - f_p \in \mathrm{H}^{1/2}$ as $p \in \mathbb{N}$, and by (1.10) we also have

$$\mathrm{D}[\mathrm{P}[f-f_p]] \to 0 \quad \text{as } p \to \infty\,. \tag{2.13}$$

Thus setting $\boldsymbol{f} \coloneqq [f/ \doteqdot]$, we see that $\boldsymbol{f} - \boldsymbol{f}_p \in \boldsymbol{H}$ as $p \in \mathbb{N}$, and by (2.13) and (2.2) we obtain

$$2\|\boldsymbol{f} - \boldsymbol{f}_p\|_{\boldsymbol{H}}^2 = \mathrm{D}[\mathrm{P}[f-f_p]] \to 0 \quad \text{as } p \to \infty\,. \tag{2.14}$$

Since $\boldsymbol{B}_\gamma$ is a linear and continuous operator in $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$, we conclude from (2.14) that

$$\|\boldsymbol{B}_\gamma(\boldsymbol{f} - \boldsymbol{f}_p)\|_{\boldsymbol{H}} = \|\boldsymbol{B}_\gamma(\boldsymbol{f}) - \boldsymbol{B}_\gamma(\boldsymbol{f}_p)\|_{\boldsymbol{H}} \to 0 \quad \text{as } p \to \infty\,. \tag{2.15}$$

On the other hand,

$$\mathrm{P}_\gamma[f_p](z) = \hat{f}_p(0;\gamma) + \sum_{n=1}^{\infty} \hat{f}_p(n;\gamma) z^n + \sum_{n=1}^{\infty} \hat{f}_p(-n;\gamma)\overline{z}^n\,, \quad p \in \mathbb{N}, \tag{2.16}$$

where by (1.11),

$$\hat{f}_p(n;\gamma) = \sum_{m=-\infty}^{\infty} \hat{\gamma}(m,n)\hat{f}_p(m) = \sum_{m=-p}^{p} \hat{\gamma}(m,n)\hat{f}(m)\,,\ n \in \mathbb{Z},\ p \in \mathbb{N}. \tag{2.17}$$

By (2.13), Corollary 1.1 and Remark 1.2 we have

$$\mathrm{D}[\mathrm{P}_\gamma[f] - \mathrm{P}_\gamma[f_p]] = \mathrm{D}[\mathrm{P}_\gamma[f-f_p]] \le K\,\mathrm{D}[\mathrm{P}[f-f_p]] \to 0 \quad \text{as } p \to \infty\,. \tag{2.18}$$

Fix $p \in \mathbb{N}$. By (1.2), (1.3), (1.11) and (2.17) we see that for every $n \in \mathbb{Z}$,

$$\begin{aligned}
\widehat{(f_p\circ\gamma)}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_p\circ\gamma(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Bigg(\sum_{m=0}^{p}\hat{f}(m)\gamma(\mathrm{e}^{\mathrm{i}t})^m + \sum_{m=1}^{p}\hat{f}(-m)\overline{\gamma(\mathrm{e}^{\mathrm{i}t})}^m\Bigg)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Bigg(\sum_{m=-p}^{p}\hat{f}(m)\gamma(\mathrm{e}^{\mathrm{i}t})^m\Bigg)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t \\
&= \sum_{m=-p}^{p}\hat{f}(m)\frac{1}{2\pi}\int_{-\pi}^{\pi}\gamma(\mathrm{e}^{\mathrm{i}t})^m\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t \\
&= \sum_{m=-p}^{p}\hat{f}(m)\hat{\gamma}(m,n) = \sum_{m=-\infty}^{\infty}\hat{\gamma}(m,n)\hat{f}_p(m) = \hat{f}_p(n;\gamma)\,.
\end{aligned} \tag{2.19}$$

Write $\boldsymbol{f} := [f/\doteqdot]$ and choose $g \in \boldsymbol{B}_\gamma(\boldsymbol{f})$. By (2.11), (2.15) and (2.2) we have

$$\mathrm{D}[\mathrm{P}[g]-\mathrm{P}[f_p\circ\gamma]] = \mathrm{D}[\mathrm{P}[g-f_p\circ\gamma]] = 2\|\boldsymbol{B}_\gamma(\boldsymbol{f})-\boldsymbol{B}_\gamma(\boldsymbol{f}_p)\|^2_{\boldsymbol{H}} \to 0 \quad \text{as } p\to\infty\,.$$

Combining this with (1.8), (2.19) and (2.18), we obtain

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}|n||\hat{f}(n;\gamma)-\hat{g}(n)|^2 &= \sum_{n=-\infty}^{\infty}|n||\hat{f}(n;\gamma)-\hat{f}_p(n;\gamma)+\hat{f}_p(n;\gamma)-\hat{g}(n)|^2 \\
&\le \sum_{n=-\infty}^{\infty}2|n|\big(|\hat{f}(n;\gamma)-\hat{f}_p(n;\gamma)|^2+|\hat{f}_p(n;\gamma)-\hat{g}(n)|^2\big) \\
&= 2\sum_{n=-\infty}^{\infty}|n||\hat{f}(n;\gamma)-\hat{f}_p(n;\gamma)|^2 + 2\sum_{n=-\infty}^{\infty}|n||\widehat{(f_p\circ\gamma)}(n)-\hat{g}(n)|^2 \\
&= \frac{1}{\pi}\,\mathrm{D}[\mathrm{P}_\gamma[f]-\mathrm{P}_\gamma[f_p]] + \frac{1}{\pi}\,\mathrm{D}[\mathrm{P}[f_p\circ\gamma]-\mathrm{P}[g]] \to 0 \quad \text{as } p\to\infty\,.
\end{aligned}$$

Hence

$$\sum_{n=-\infty}^{\infty}|n||\hat{f}(n;\gamma)-\hat{g}(n)|^2 = 0\,,$$

and so $\hat{f}(n;\gamma) = \hat{g}(n)$ for $n \in \mathbb{Z}\setminus\{0\}$. This means that $\mathrm{P}[g]-\mathrm{P}_\gamma[f]$ is a constant function in $\mathbb{D}$. To be more precise,

$$\mathrm{P}[g](z)-\mathrm{P}_\gamma[f](z) = \mathrm{P}[g](0)-\mathrm{P}_\gamma[f](0) = \hat{g}(0)-\hat{f}(0;\gamma) = \hat{g}(0)-\hat{f}(0)\,,\ z\in\mathbb{D}.$$

Finally, $\mathrm{Tr}[\mathrm{P}[g]]-\mathrm{Tr}[\mathrm{P}_\gamma[f]] = \hat{g}(0)-\hat{f}(0)$ a.e. in $\mathbb{T}$, i.e. $g \doteqdot \mathrm{Tr}[\mathrm{P}_\gamma[f]]$. This implies the property (2.6), which is the desired conclusion. □

Corollary 2.2. *For every $\gamma \in \mathrm{Q}(\mathbb{T})$,*

$$\mathrm{D}[\mathrm{P}_\gamma[f]] = 2\|\boldsymbol{B}_\gamma([f/\div])\|^2_{\boldsymbol{H}}\,, \quad f \in \operatorname{Re}\mathrm{H}^{1/2}\,. \tag{2.20}$$

Proof. By Theorem 2.1 and (2.2) we conclude that for every $f \in \operatorname{Re}\mathrm{H}^{1/2}$,

$$2\|\boldsymbol{B}_\gamma([f/\div])\|^2_{\boldsymbol{H}} = 2\|[\mathrm{Tr}[\mathrm{P}_\gamma[f]]/\div]\|^2_{\boldsymbol{H}} = \mathrm{D}[\mathrm{P}[\mathrm{Tr}[\mathrm{P}_\gamma[f]]]] = \mathrm{D}[\mathrm{P}_\gamma[f]]\,,$$

which yields (2.20). □

3. The smallest positive eigenvalue of a quasisymmetric automorphism of the unit circle. The operator P_γ seems to be a more convenient tool for studying spectral values and eigenvalues problems as compared to the operator $\boldsymbol{B}_\gamma$. Applying Theorem 2.1 and Corollary 2.2, we can rewrite a number of known so far results in this subject by means of the operator P_γ. As an example we will show Theorem 3.1 which is related to the following Krzyż result on quasiconformal reflection; cf. [10].

Theorem B. *Let $\Gamma \subset \mathbb{C}$ be a quasicircle and let $F : \mathrm{cl}(\Omega) \to \mathbb{C}$ and $F_* : \mathrm{cl}(\Omega_*) \to \mathbb{C}$ be continuous and locally univalent functions on $\mathrm{cl}(\Omega)$ and $\mathrm{cl}(\Omega_*)$, and analytic in the complementary domains Ω and $\Omega_* \ni \infty$ of Γ, respectively. Assume that both the functions F and F_* have finite Dirichlet integrals on Ω and Ω_*, respectively. If the equalities (0.3) hold on Γ with a real constant λ, then $|\lambda| > 1$, Γ admits a unique extremal K-quasiconformal (i.e. with the smallest maximal dilatation K) reflection Ψ with $K = (|\lambda|+1)/(|\lambda|-1)$, and the following equality holds*

$$F(z) = \frac{F_*\circ\Psi(z) + \lambda\overline{F_*\circ\Psi(z)}}{1-\lambda}\,, \quad z \in \mathrm{cl}(\Omega)\,. \tag{3.1}$$

Let $\mathrm{S}[f]$ be the Schwarz integral of a function $f \in \mathrm{L}^1(\mathbb{T})$, i.e.

$$\mathrm{S}[f](z) \coloneqq \frac{1}{2\pi}\int_{\mathbb{T}} f(u)\frac{u+z}{u-z}|\mathrm{d}u| = \hat{f}(0) + 2\sum_{n=1}^{\infty}\hat{f}(n)z^n\,, \quad z \in \mathbb{D}. \tag{3.2}$$

Since the kernel function is holomorphic with respect to z, $\mathrm{S}[f]$ is a holomorphic function. Moreover, by (3.2) and (1.1) we see that $\mathrm{P}[f] = \operatorname{Re}\mathrm{S}[f]$ as $f \in \operatorname{Re}\mathrm{L}^1(\mathbb{T})$, and hence $\operatorname{Im}\mathrm{S}[f]$ is a harmonic conjugate function to $\mathrm{P}[f]$ and $\operatorname{Im}\mathrm{S}[f](0) = 0$. Therefore, $\operatorname{Re}\mathrm{L}^1(\mathbb{T}) \ni f \mapsto \mathrm{Tr}[\operatorname{Im}\mathrm{S}[f]]$ is called the *harmonic conjugation operator*; cf. e.g. [4, Sect. III.1]. Since $\mathrm{D}[\operatorname{Re}F] = \mathrm{D}[\operatorname{Im}F]$ for any holomorphic function F in $\mathbb{D}$, we conclude from the definition of the space $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$ that the operator $\boldsymbol{A}$, defined by

$$\boldsymbol{A}([f/\div]) \coloneqq [\mathrm{Tr}[\operatorname{Im}\mathrm{S}[f]]/\div]\,, \quad f \in \operatorname{Re}\mathrm{H}^{1/2}, \tag{3.3}$$

satisfies the following properties

$$\boldsymbol{A}(\boldsymbol{H}) = \boldsymbol{H}, \quad \boldsymbol{A}^2(\boldsymbol{f}) = -\boldsymbol{f} \quad \text{and} \quad \|\boldsymbol{A}(\boldsymbol{f})\|_{\boldsymbol{H}} = \|\boldsymbol{f}\|_{\boldsymbol{H}}, \quad \boldsymbol{f} \in \boldsymbol{H}, \tag{3.4}$$

and so the operator $\boldsymbol{A}$ is an isometry of the space $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$ onto itself. Therefore, for each $\gamma \in \mathrm{Q}(\mathbb{T})$ the operator

$$\boldsymbol{A}_\gamma \coloneqq \boldsymbol{B}_\gamma \boldsymbol{A} \boldsymbol{B}_\gamma^{-1}, \tag{3.5}$$

called the *generalized harmonic conjugation operator*, is a linear homeomorphism of the space $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$ onto itself; cf. [24, Sect. 3.1]. We recall that a real number λ is said to be an *eigenvalue of* $\gamma \in \mathrm{Q}(\mathbb{T})$ if there exists $\boldsymbol{f} \in \boldsymbol{H}$ with $\|\boldsymbol{f}\|_{\boldsymbol{H}} \neq 0$ such that

$$(\lambda + 1)\boldsymbol{A}(\boldsymbol{f}) = (\lambda - 1)\boldsymbol{A}_\gamma(\boldsymbol{f})\,; \tag{3.6}$$

cf. [21, Definition 1.1] and also [24, Sect. 3.2]. Let Λ_γ^* be the set of all eigenvalues of $\gamma \in \mathrm{Q}(\mathbb{T})$. We recall that a quasiconformal self-mapping ψ of $\mathbb{D}$ is said to be a *regular Teichmüller mapping* if there exists a non-zero holomorphic function F in $\mathbb{D}$ and a constant k, $0 \leq k < 1$, such that the complex dilatation of ψ is of the form

$$\frac{\bar{\partial}\psi}{\partial\psi} = k\frac{\overline{F}}{|F|} \quad \text{a.e. in } \mathbb{D}.$$

Under the assumptions of Theorem B, let $H : \mathrm{cl}(\mathbb{D}) \to \mathrm{cl}(\Omega)$ and $H_* : \mathrm{cl}(\mathbb{D}_*) \to \mathrm{cl}(\Omega_*)$ be continuous mappings and conformal in $\mathbb{D}$ and $\mathbb{D}_*$, respectively. Then $\gamma \coloneqq H_*^{-1} \circ H$ is a sense-preserving homeomorphic self-mapping of $\mathbb{T}$. Let $\hbar$ be the mapping defined by

$$\hbar(z) \coloneqq 1/z \text{ as } z \in \mathbb{C} \setminus \{0\} \quad \text{and} \quad \hbar(0) \coloneqq \infty\,,\ \hbar(\infty) \coloneqq 0\,.$$

If Γ admits a Q-quasiconformal reflection Ψ, then $\psi \coloneqq \overline{\hbar} \circ H_*^{-1} \circ \Psi \circ H$ is a Q-quasiconformal extension of γ to $\mathbb{D}$. Conversely, if ψ is a Q-quasiconformal extension of γ to $\mathbb{D}$, then the mapping Ψ defined by

$$\Psi(z) \coloneqq H_* \circ \overline{\hbar} \circ \psi \circ H^{-1}(z) \text{ as } z \in \mathrm{cl}(\Omega) \quad \text{and} \quad \Psi(z) \coloneqq \Psi^{-1}(z) \text{ as } z \in \Omega_*\,,$$

is a Q-quasiconformal reflection in Γ. Thus for every $Q \geq 1$, Γ admits a Q-quasiconformal reflection iff $\gamma \in \mathrm{Q}(\mathbb{T}, Q)$. In particular, since Ψ is an extremal K-quasiconformal reflection, ψ is an extremal K-quasiconformal extension of γ to $\mathbb{D}$. Moreover, by (3.1), the complex dilatation of ψ satisfies

$$\frac{\bar{\partial}\psi}{\partial\psi} = \overline{\left[\frac{(\partial\Psi \circ H)H'}{(\bar{\partial}\Psi \circ H)\overline{H'}}\right]} = -\frac{1}{\lambda}\frac{\overline{(F' \circ H)H'}}{(F' \circ H)H'} = -\frac{1}{\lambda}\frac{\overline{(F \circ H)'}}{(F \circ H)'} \quad \text{a.e. in } \mathbb{D},$$

i.e. ψ is a regular Teichmüller mapping. From [24, Theorems 4.5.2 and 4.4.2] it follows that λ is an eigenvalue of γ. Therefore, Theorem B is deeply related to the following theorem.

Theorem 3.1. *Let $\gamma \in \mathrm{Q}(\mathbb{T}) \setminus \mathrm{Q}(\mathbb{T}, 1)$. Then the following properties are equivalent to each other:*

(i) *There exists* $f \in \mathrm{H}^{1/2}$ *such that*

$$\mathrm{D}[\mathrm{P}_\gamma[f]] = \mathrm{K}(\gamma)\,\mathrm{D}[\mathrm{P}[f]] \neq 0\,, \tag{3.7}$$

where $\mathrm{K}(\gamma) := \inf\bigl(\{K \geq 1 : \gamma \in \mathrm{Q}(\mathbb{T}, K)\}\bigr)$*;*

(ii) *There exists the smallest positive eigenvalue of* γ *and*

$$\min(\{\lambda \in \Lambda^*_\gamma : \lambda > 0\}) = (\mathrm{K}(\gamma)+1)/(\mathrm{K}(\gamma)-1)\,; \tag{3.8}$$

(iii) *There exists a holomorphic function* $F : \mathbb{D} \to \mathbb{C}$ *such that* $0 \neq \mathrm{D}[F] < +\infty$ *and* γ *admits a regular quasiconformal Teichmüller extension* ψ *to* $\mathbb{D}$ *with the complex dilatation*

$$\frac{\bar\partial\psi}{\partial\psi} = \frac{\mathrm{K}(\gamma)-1}{\mathrm{K}(\gamma)+1}\frac{\overline{F'}}{F'} \quad \textit{a.e. in } \mathbb{D}. \tag{3.9}$$

Proof. Let $\gamma \in \mathrm{Q}(\mathbb{T}) \setminus \mathrm{Q}(\mathbb{T}, 1)$ be arbitrarily fixed.

Assume first that the property (i) holds. Since $f = \operatorname{Re} f + \mathrm{i}\operatorname{Im} f$, we deduce from (1.7) that

$$\mathrm{D}[\mathrm{P}[\operatorname{Re} f]] + \mathrm{D}[\mathrm{P}[\operatorname{Im} f]] = \mathrm{D}[\mathrm{P}[f]]\,, \tag{3.10}$$

as well as, by Remark 1.2,

$$\mathrm{D}[\mathrm{P}_\gamma[\operatorname{Re} f]] + \mathrm{D}[\mathrm{P}_\gamma[\operatorname{Im} f]] = \mathrm{D}[\mathrm{P}_\gamma[f]] = \mathrm{K}(\gamma)\,\mathrm{D}[\mathrm{P}[f]] \neq 0\,. \tag{3.11}$$

Suppose that $\mathrm{D}[\mathrm{P}_\gamma[\operatorname{Re} f]] < \mathrm{K}(\gamma)\mathrm{D}[\mathrm{P}[\operatorname{Re} f]]$ or $\mathrm{D}[\mathrm{P}_\gamma[\operatorname{Im} f]] < \mathrm{K}(\gamma)\mathrm{D}[\mathrm{P}[\operatorname{Im} f]]$. Then by (3.10) we see that

$$\begin{aligned}\mathrm{D}[\mathrm{P}_\gamma[\operatorname{Re} f]] + \mathrm{D}[\mathrm{P}_\gamma[\operatorname{Im} f]] &< \mathrm{K}(\gamma)\,\mathrm{D}[\mathrm{P}[\operatorname{Re} f]] + \mathrm{K}(\gamma)\,\mathrm{D}[\mathrm{P}[\operatorname{Im} f]] \\ &= \mathrm{K}(\gamma)\,\mathrm{D}[\mathrm{P}[f]]\,,\end{aligned}$$

which contradicts (3.11). Therefore,

$$\mathrm{D}[\mathrm{P}_\gamma[\operatorname{Re} f]] = \mathrm{K}(\gamma)\,\mathrm{D}[\mathrm{P}[\operatorname{Re} f]] \quad \text{and} \quad \mathrm{D}[\mathrm{P}_\gamma[\operatorname{Im} f]] = \mathrm{K}(\gamma)\,\mathrm{D}[\mathrm{P}[\operatorname{Im} f]]\,.$$

From (3.11) it follows that $\mathrm{D}[\mathrm{P}_\gamma[\operatorname{Re} f]] \neq 0$ or $\mathrm{D}[\mathrm{P}_\gamma[\operatorname{Im} f]] \neq 0$. Without loss of generality we may assume that the first possibility holds. Then setting $\boldsymbol{g} := [\operatorname{Re} f/ \doteqdot]$, we conclude from Corollary 2.2 and (2.2) that

$$\|\boldsymbol{B}_\gamma(\boldsymbol{g})\|^2_{\boldsymbol{H}} = \frac{1}{2}\,\mathrm{D}[\mathrm{P}_\gamma[\operatorname{Re} f]] = \frac{1}{2}\,\mathrm{K}(\gamma)\,\mathrm{D}[\mathrm{P}[\operatorname{Re} f]] = \mathrm{K}(\gamma)\|\boldsymbol{g}\|^2_{\boldsymbol{H}} \neq 0\,,$$

and hence $\boldsymbol{f} := \boldsymbol{g}/\|\boldsymbol{g}\|_{\boldsymbol{H}}$ satisfies the equality

$$\|\boldsymbol{B}_\gamma(\boldsymbol{f})\|^2_{\boldsymbol{H}} = \mathrm{K}(\gamma)\,. \tag{3.12}$$

Applying now [23, Theorem 2.1 ($\Leftarrow$)], we see that the property (ii) holds, which proves the implication (i) $\Rightarrow$ (ii). Applying next [23, Theorem 1.3 ($\Leftarrow$)], we deduce from (3.12) the property (iii), and so the implication (i) $\Rightarrow$ (iii) is true.

Conversely, suppose that the property (ii) holds. Then by [23, Theorem 2.1 ($\Rightarrow$)] we see that the equality (3.12) holds for a certain $\boldsymbol{f} \in \boldsymbol{H}$ such that $\|\boldsymbol{f}\|_{\boldsymbol{H}} = 1$. The same conclusion can be deduced from [23, Theorem

1.3 (⇒)] provided the property (iii) holds. Applying now Corollary 2.2 and (2.2), we obtain

$$\mathrm{D}[\mathrm{P}_\gamma[f]] = 2\|\boldsymbol{B}_\gamma(\boldsymbol{f})\|^2_{\boldsymbol{H}} = 2\,\mathrm{K}(\gamma)\|\boldsymbol{f}\|^2_{\boldsymbol{H}} = \mathrm{K}(\gamma)\,\mathrm{D}[\mathrm{P}[f]] \neq 0\ , \quad f \in \boldsymbol{f}\ ,$$

which leads to (3.7). Therefore, the implications (ii) ⇒ (i) and (iii) ⇒ (i) are true, which completes the proof. □

Remark 3.2. It turns out that the equivalence (i) ⇔ (iii) in Theorem 3.1 can be improved as follows.
Suppose that $\gamma \in \mathrm{Q}(\mathbb{T})$ *and* $f \in \mathrm{H}^{1/2}$ *satisfy* $\mathrm{K}(\gamma) > 1$ *and* $\mathrm{D}[\mathrm{P}[f]] > 0$. *Then*

$$\mathrm{D}[\mathrm{P}_\gamma[f]] = \mathrm{K}(\gamma)\,\mathrm{D}[\mathrm{P}[f]]$$

if and only if there exist $\alpha, c \in \mathbb{R}$ *such that* $\mathrm{e}^{\mathrm{i}\alpha} f(z) - c\mathrm{i} \in \mathbb{R}$ *for a.e.* $z \in \mathbb{T}$ *and* γ *admits a regular quasiconformal Teichmüller extension* ψ *to* $\mathbb{D}$ *with the complex dilatation*

$$\frac{\bar\partial\psi}{\partial\psi} = \mathrm{e}^{-2\mathrm{i}\alpha}\frac{\mathrm{K}(\gamma)-1}{\mathrm{K}(\gamma)+1}\frac{\overline{\partial\,\mathrm{P}_\gamma[f]}}{\partial\,\mathrm{P}_\gamma[f]} \quad \textit{a.e. in } \mathbb{D}.$$

However, the proof of this statement exceeds the scope of this paper and will be published elsewhere.

Remark 3.3. The smallest positive eigenvalue λ of γ in Theorem 3.1 is strictly related to a number of important constants like, e.g.: the Schober constant, the Grunsky–Kühnau constant, as well as the supremum norms of the Neumann–Poincaré operator, the Hilbert transformation and the operator $\boldsymbol{A}_\gamma$. For the detailed exposition of this topic the reader is referred to [24, Sect. 4.4]. By [24, Theorem 4.4.2], the set $\Lambda^*(\Gamma)$ of all eigenvalues of a quasicircle $\Gamma \subset \mathbb{C}$ coincides with the set Λ^*_γ, where γ is a welding homeomorphism of Γ. Therefore, Theorem 3.1 is closely related to various results involving these constants obtained by Kühnau ([15], [16], [17]), Schiffer ([29]) and Krushkal ([6], [7], [8, Sect. 2]); see also the survey article by Krushkal [9, p. 528] and the references given there. This relationship provides a motivation for the further study of the inequalities (1.6) in Theorem A.

References

[1] Beurling, A., Ahlfors, L. V., *The boundary correspondence under quasiconformal mappings*, Acta Math. **96** (1956), 125–142.
[2] Duren, P., *Theory of H^p-Spaces*, Dover Publications, Inc., Mineola, New York, 2000.
[3] Gaier, D., *Konstruktive Methoden der konformen Abbildung*, Springer-Verlag, Berlin, 1964.
[4] Garnett, J. B., *Bounded Analytic Functions*, Academic Press, New York, 1981.
[5] Kellogg, O. D., *Foundations of Potential Theory*, Dover Publications, Inc., New York, 1953.
[6] Krushkal, S. L., *On the Grunsky coefficient conditions*, Siberian Math. J. **28** (1987), 104–110.

[7] Krushkal, S. L., *Grunsky coefficient inequalities, Carathéodory metric and extremal quasiconformal mappings*, Comment. Math. Helv. **64** (1989), 650–660.

[8] Krushkal, S. L., *Univalent holomorphic functions with quasiconformal extensions (variational approach)*, Handbook of Complex Analysis: Geometric Function Theory. Vol. 2 (ed. by R. Kühnau), Elsevier B.V., 2005, pp. 165–241.

[9] Krushkal, S. L., *Quasiconformal Extensions and Reflections*, Handbook of Complex Analysis: Geometric Function Theory. Vol. 2 (ed. by R. Kühnau), Elsevier B.V., 2005, pp. 507–553.

[10] Krzyż, J. G., *Conjugate holomorphic eigenfunctions and extremal quasiconformal reflection*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 305–311.

[11] Krzyż, J. G., *Generalized Fredholm eigenvalues of a Jordan curve*, Ann. Polon. Math. **46** (1985), 157–163.

[12] Krzyż, J. G., *Quasicircles and harmonic measure*, Ann. Acad. Sci. Fenn. Ser. A I Math. **12** (1987), 19–24.

[13] Krzyż, J. G., *Quasisymmetric functions and quasihomographies*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **47** (1993), 90–95.

[14] Krzyż, J. G., Partyka, D., *Generalized Neumann-Poincaré operator, chord-arc curves and Fredholm eigenvalues*, Complex Variables Theory Appl. **21** (1993), 253–263.

[15] Kühnau, R., *Zu den Grunskyschen Coeffizientenbedingungen*, Ann. Acad. Sci. Fenn. Ser. A I Math. **6** (1981), 125–130.

[16] Kühnau, R., *Quasikonforme Fortsetzbarkeit, Fredholmsche Eigenwerten und Grunskysche Koeffizientenbedingungen*, Ann. Acad. Sci. Fenn. Ser. A I Math. **7** (1982), 383–391.

[17] Kühnau, R., *Wann sind die Grunskyschen Koeffizientenbedingungen hinreichend für Q-quasikonforme Fortsetzbarkeit?*, Comment. Math. Helv. **61** (1986), 290–307.

[18] Kühnau, R., *Koeffizientenbedingungen vom Grunskyschen Typ und Fredholmsche Eigenwerte*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **58** (2004), 79–87.

[19] Kühnau, R., *A new matrix characterization of Fredholm eigenvalues of quasicircles*, J. Anal. Math. **99** (2006), 295–307.

[20] Kühnau, R., *New characterizations of Fredholm eigenvalues of quasicircles*, Rev. Roumaine Math. Pures Appl. **51** (2006), no. 5–6, 683–688.

[21] Partyka, D., *Spectral values and eigenvalues of a quasicircle*, Ann. Univ. Mariae Curie-Skłodowska Sec. A **46** (1993), 81–98.

[22] Partyka, D., *The smallest positive eigenvalue of a quasisymmetric automorphism of the unit circle*, Topics in Complex Analysis (Warsaw, 1992), Banach Center Publ., 31, Polish Acad. Sci., Warsaw, 1995, pp. 303–310.

[23] Partyka, D., *Some extremal problems concerning the operator B_γ*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **50** (1996), 163–184.

[24] Partyka, D., *The generalized Neumann–Poincaré operator and its spectrum*, Dissertationes Math. (Rozprawy Mat.) **366** (1997), 125 pp.

[25] Partyka, D., *Eigenvalues of quasisymmetric automorphisms determined by VMO functions*, Ann. Univ. Mariae Curie-Skłodowska Sec. A **52** (1998), 121–135.

[26] Partyka, D., *The Grunsky type inequalities for quasisymmetric automorphisms of the unit circle*, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. **31** (2000), 135–142.

[27] Partyka, D., Sakan, K., *A conformally invariant dilatation of quasisymmetry*, Ann. Univ. Mariae Curie-Skłodowska Sec. A **53** (1999), 167–181.

[28] Pommerenke, Ch., *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

[29] Schiffer, M., *Fredholm eigenvalues and Grunsky matrices*, Ann. Polon. Math. **39** (1981), 149–164.

[30] Schober, G., *Numerische, insbesondere approximationstheoretische behandlung von funktionalgleichungen*, Estimates for Fredholm Eigenvalues Based on Quasiconformal Mapping, Lecture Notes in Math. 333, Springer-Verlag, Berlin, 1973, pp. 211–217.
[31] Shen, Y., *Generalized Fourier coefficients of a quasi-symmetric homeomorphism and Fredholm eigenvalue*, J. Anal. Math. **112** (2010), no. 1, 33–48.
[32] Warschawski, S. E., *On differentiability at the boundary in conformal mapping*, Proc. Amer. Math. Soc. **12** (1961), 614–620.
[33] Zając, J., *Quasihomographies in the theory of Teichmüller spaces*, Dissertationes Math. (Rozprawy Mat.) **357** (1996), 102 pp.

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