doi: 10.2478/v10062-010-0010-1

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXIV, NO. 2, 2010 SECTIO A 15–19

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On the real X-ranks of points of $\mathbb{P}^n(\mathbb{R})$ with respect to a real variety $X \subset \mathbb{P}^n$

Abstract. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety defined over $\mathbb{R}$. For any $P \in \mathbb{P}^n(\mathbb{R})$ the real X-rank $r_{X,\mathbb{R}}(P)$ is the minimal cardinality of $S \subset X(\mathbb{R})$ such that $P \in \langle S \rangle$. Here we extend to the real case an upper bound for the X-rank due to Landsberg and Teitler.

1. Introduction. Fix an integral and non-degenerate variety $X \subseteq \mathbb{P}^n$ defined over $\mathbb{C}$. For any $P \in \mathbb{P}^n(\mathbb{C})$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X(\mathbb{C})$ such that $P \in \langle S \rangle$, where $\langle \ \ \rangle$ denote the linear span. Hence $r_X(P) = 1$ if and only if $P \in X(\mathbb{C})$. Since X is non-degenerate, the X-ranks are defined and $r_X(P) \leq n+1$ for all $P \in \mathbb{P}^n(\mathbb{C})$. As a motivation for the study of X-ranks, see [1], [5], [7], [9], [11] and references therein. Now assume that X is defined over $\mathbb{R}$ and that the embedding $X \subset \mathbb{P}^n$ is defined over $\mathbb{R}$, i.e. the scheme X is cut out inside $\mathbb{P}^n$ by homogeneous polynomials with real coefficients. For any $P \in \mathbb{P}^n(\mathbb{R})$ the real X-rank $r_{X,\mathbb{R}}(P)$ is the minimal cardinality of a finite set $S \subset X(\mathbb{R})$ such that $P \in \langle S \rangle$, with the convention $r_{X,\mathbb{R}}(P) = +\infty$ if no such set exists. Notice that $r_{X,\mathbb{R}}(P) = +\infty$ if and only if $P \notin \langle X(\mathbb{R}) \rangle$. Hence the function $r_{X,\mathbb{R}}$ is integer-valued if and only if the set $X(\mathbb{R})$ spans $\mathbb{P}^n$. Notice that if $r_{X,\mathbb{R}}(P) \neq +\infty$, then $r_{X,\mathbb{R}}(P) \leq n+1$. Now assume that the smooth quasi-projective variety X_{reg} has real points, i.e. assume

2000 *Mathematics Subject Classification.* 14N05, 14H50.
Key words and phrases. Ranks, real variety, structured rank.
The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

$X_{reg}(\mathbb{R}) \neq \emptyset$. Thus around P the set $X(\mathbb{R})$ contains a smooth real algebraic manifold of dimension m. Since X is irreducible, we get that $X_{reg}(\mathbb{R})$ is Zariski dense in $X(\mathbb{C})$. Since $X(\mathbb{C})$ spans $\mathbb{P}^n$, $\langle X(\mathbb{R})\rangle = \mathbb{P}^n$ if $X_{reg}(\mathbb{R}) \neq \emptyset$. If $X_{reg}(\mathbb{R}) = \emptyset$, then $X(\mathbb{R})$ is contained in a proper Zariski closed subset $\mathrm{Sing}(X)$ of X. Quite often $\langle \mathrm{Sing}(X)\rangle \neq \mathbb{P}^n$ even when $\mathrm{Sing}(X) \neq \emptyset$. If X is a reduced curve, then $X_{reg}(\mathbb{R}) \neq \emptyset$ if and only if the set $X(\mathbb{R})$ is infinite.

We prove the following extension of [11], Proposition 5.1, under the assumption $X_{reg}(\mathbb{R}) \neq \emptyset$.

Theorem 1. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety defined over $\mathbb{R}$. Set $d := \deg(X)$. Assume $X_{reg}(\mathbb{R}) \neq \emptyset$. Then:*

(i) *$r_{X,\mathbb{R}}(P) \leq n+2-m$ for all $P \in \mathbb{P}^n(\mathbb{R})$.*

(ii) *If $d-m+1 \equiv n \pmod 2$, then $r_{X,\mathbb{R}}(P) \leq n+1-m$ for all $P \in \mathbb{P}^n(\mathbb{R})$.*

By [11], Proposition 5.1, we have $r_X(P) \leq n+1-m$ for all $P \in \mathbb{P}^n$ and this bound is in general sharp. Moreover, the most important case in which the upper bound $r_X(P) = n+1-m$ is reached is defined over $\mathbb{R}$, it is smooth and with non-empty real locus: the rational normal curve of $\mathbb{P}^n$ ([8] or [11], Theorem 4.1). Hence the bound in part (ii) of Theorem 1 cannot be improved without making additional assumptions on the variety X. See Example 1 for a case in which equality holds in part (i) of Theorem 1.

Our proof of Theorem 1 is just an adaptation of the proof of [11], Proposition 5.1.

The interested reader may find related topics in [3] (definition of the X-K-rank $r_{X,K}(P)$ for an arbitrary field K and some computations of it when X is a rational normal curve), and in [4], Proposition 3 (subsets of $X(K)$ computing the integer $r_{X,K}(P)$ when X is a rational normal curve).

2. Proof of Theorem 1 and an example.

Lemma 1. *Let $X \subset \mathbb{P}^2$ be an integral curve of even degree d defined over $\mathbb{R}$. Assume $X_{reg}(\mathbb{R}) \neq \emptyset$. Then $r_{X,\mathbb{R}}(P) \leq 2$ for all $P \in \mathbb{P}^2(\mathbb{R})$.*

Proof. If $P \in X(\mathbb{R})$, then $r_{X,\mathbb{R}}(P) = 1$. Fix any $P \in \mathbb{P}^2(\mathbb{R})\backslash X(\mathbb{R})$. Since we work in characteristic zero, X is not a strange curve ([10] Ex. IV.3.8). Thus there is a non-empty open subset E of $X_{reg}(\mathbb{C})$ such that $P \notin T_QX$ for all $Q \in E$. Since $X_{reg}(\mathbb{R}) \neq \emptyset$, the set $X_{reg}(\mathbb{R})$ is Zariski dense in $X(\mathbb{C})$. Hence there is $Q \in E \cap X_{reg}(\mathbb{R})$. Thus the line $D := \langle\{P, Q\}\rangle$ intersects transversally X at Q. Since d is even, the line D must contain another point of $X(\mathbb{R})$. Thus $r_{X,\mathbb{R}}(P) \leq 2$. □

Proof of Theorem 1. The proof of the reduction of the case "$m \geq 2$" to the case "$m = 1$" is an easy adaption of the proof given by Landsberg and Teitler over $\mathbb{C}$. Only the case $m = 1$ gives a small surprise.

(a) Here we assume $m = 1$. If $d - n$ is odd, then there is nothing to prove, because $X_{reg}(\mathbb{R})$ spans $\mathbb{P}^n$. Hence we may assume $d \equiv n \pmod 2$.

We use induction on n. If $n = 2$, then apply Lemma 1. Now assume $n \geq 3$. Fix a general $Q \in X(\mathbb{C})$. Hence X is smooth at Q. Thus the linear projection $\ell_Q : \mathbb{P}^n \backslash \{Q\} \to \mathbb{P}^{n-1}$ induces a morphism $v_Q : X \to \mathbb{P}^{n-1}$ such that $\deg(v_Q) \cdot \deg(v_Q(X)) = d - 1$. In characteristic zero a general secant line of X is not a multisecant line. Hence for a general Q we have $\deg(v_Q) = 1$, i.e., the curve $v_Q(X)$ is an integral and non-degenerate subcurve of $\mathbb{P}^{n-1}$ with degree $d - 1$. Since $X_{reg}(\mathbb{R})$ is Zariski dense in $X_{reg}(\mathbb{R})$, this is true also for almost all (except at most finitely many) points $Q \in X_{reg}(\mathbb{R})$. Fix $Q \in X_{reg}(\mathbb{R})$ such that $\deg(v_Q) = 1$. Thus $T := v_Q(X) \subset \mathbb{P}^{n-1}$ is an integral and non-degenerate curve defined over $\mathbb{R}$ and such that $T_{reg}(\mathbb{R}) \neq \emptyset$. Since $d-1 \equiv n-1 \pmod 2$, the inductive assumption gives $r_{T,\mathbb{R}}(v_Q(P)) \leq n-1$. This is not sufficient to conclude that $r_{X,\mathbb{R}}(P) \leq n$, because $v_P(X)(\mathbb{R})$ may be larger than $v_P(X(\mathbb{R}))$. However, we may adapt the proof of Lemma 1 in the following way. Fix a general $(Q_1, \dots, Q_{n-2}) \in X(\mathbb{C})^{(n-2)}$. Hence X is smooth at each Q_i. Set $U := \langle Q_1, \dots, Q_{n-2} \rangle$. Since the points $Q_1, \dots, Q_{n-2}$ are general and X is non-degenerate, $\dim(U) = n - 3$. Since we are in characteristic zero, a general hyperplane section of X is in linearly general position ([2], p. 109). Hence $X \cap U = \{Q_1, \dots, Q_{n-2}\}$ (scheme-theoretic intersection). Since $X(\mathbb{R})$ is Zariski dense in $X(\mathbb{C})$, we may find $Q_i \in X(\mathbb{R})$ with the same property. Let $\ell_U : \mathbb{P}^n \backslash U \to \mathbb{P}^2$ denote the linear projection from U. Since $X \cap U = \{Q_1, \dots, Q_{n-2}\}$ (scheme-theoretically) and $Q_i \in X_{reg}$ for all i, the map $\ell_U | (X \backslash X \cap U)$ induces a birational morphism $v_U : X \to \mathbb{P}^2$ such that $\deg(v_U(X)) = d - n + 2$ is even. The morphism v_U is defined over $\mathbb{R}$. For a general $Q_{n-1} \in X(\mathbb{R})$ the line $\langle \{v_U(P), v_U(Q_{n-1})\} \rangle$ intersects transversally $v_U(X)$ at $v_U(Q_{n-1})$. Since $\deg(v_U(X))$ is even, this line intersects $v_U(X)$ at another real point, P'. Since v_U induces a real isomorphism between the normalizations of X and of $v_U(X)$, the set $v_U(X)(\mathbb{R}) \setminus v_U(X(\mathbb{R}) \setminus U)$ is finite. Thus for a general Q_{n-1} we may assume that P' is in the image of a real point of $X \setminus U$. Hence $r_{X,\mathbb{R}}(P) \leq n$, concluding the proof in the case $m = 1$.

(b) Here we assume $m \geq 2$ and that Theorem 1 is true for varieties of dimension $m - 1$. Assume the existence of $P \in \mathbb{P}^n(\mathbb{R})$ such that $r_{X,\mathbb{R}}(P) \geq n + 2 - m$ (case $d - m + 1 \equiv 0 \pmod 2$), or $r_{X,\mathbb{R}}(P) \geq n + 1 - m$ (case $d - m + 1 \equiv 0 \pmod 2$). If $P \in X(\mathbb{R})$, then $r_{X,\mathbb{R}}(P) = 1$. Hence we may assume $P \notin X(\mathbb{R})$. Since $X(\mathbb{R}) = \mathbb{P}^n(\mathbb{R}) \cap X(\mathbb{C})$, we have $P \notin X(\mathbb{C})$. Let $A_P(\mathbb{C})$ denote the set of all hyperplanes $H \subset \mathbb{P}^n(\mathbb{C})$ containing P. The set A_P is an $(n-1)$-dimensional complex projective space. Since $P \in \mathbb{P}^n(\mathbb{R})$, the variety $A_P(\mathbb{C})$ has a real structure such that the set $A_P(\mathbb{R})$ of its real points parametrizes the set of all real hyperplanes containing P. Since $A_P(\mathbb{R})$ is Zariski dense in $A_P(\mathbb{C})$, every non-empty Zariski open subset of $A_P(\mathbb{C})$ intersects $A_P(\mathbb{R})$. Hence any non-empty open subset of $A_P(\mathbb{C})$ defined over $\mathbb{R}$ has a real point. Moreover, for a general $Q \in X(\mathbb{C})$ there is $H \in A_P(\mathbb{C})$ such that $Q \in H$. Since $X_{reg}(\mathbb{R}) \neq \emptyset$, we get the existence of $H \in A_P(\mathbb{R})$ containing a sufficiently general point of $X_{reg}(\mathbb{R})$. Hence we may find a sufficiently general $H \in A_P(\mathbb{R})$ with the additional condition $X_{reg}(\mathbb{R}) \cap H \neq \emptyset$. Bertini's theorem says that if H is general, then $X \cap H$ is an integral $(m-1)$-dimensional variety and $(X \cap H)_{reg} = X_{reg} \cap H$. Since $H \in A_P(\mathbb{R})$, the variety $X \cap H$ is defined over $\mathbb{R}$. Notice that $r_{X,\mathbb{R}}(P) \leq r_{X\cap H,\mathbb{R}}(P)$. Since $(X \cap H)_{reg}(\mathbb{R}) \neq \emptyset$ and $d - n + m \equiv d - (n-1) + (m-1) \pmod 2$, we may apply the inductive assumption to the variety $X \cap H$. □

The next example shows that the inequality in part (i) of Theorem 1 may be an equality. Hence in part (ii) of Theorem 1 the parity condition cannot be dropped without making other assumptions on X.

Example 1. Fix positive integers k, c such that $k \leq 2c$ and $(2c+1, k) = 1$. Take homogeneous coordinates x, y, z of $\mathbb{P}^2$ and set $X := \{z^{2c+1-k}y^k = x^{2c+1}\}$ and $P := (1; 0; 0)$. Hence $P \notin X$. Thus $r_{X,\mathbb{R}}(P) \geq 2$. The linear projection from P sends any $(x_0; y_0; z_0) \neq (1; 0; 0)$, onto the point $(y_0; z_0) \in \mathbb{P}^1$. Since $2c + 1$ is odd, the equation $z_0^{2c+1-k}y_0^k = t^{2c+1}$ has a unique real root. Hence $r_{X,\mathbb{R}}(P) \neq 2$. If $k = c = 1$, then the curve X is an integral plane cubic with an ordinary cusp. Taking cones we get examples with arbitrary m and $n = m + 1$.

Fix a field K and an integral and non-degenerate subvariety $X \subset \mathbb{P}^n$. Assume that both X and the embedding $X \hookrightarrow \mathbb{P}^n$ are defined over K. For each $P \in \mathbb{P}^n(K)$ the X-K-rank $r_{X,K}(P)$ of P is the minimal cardinality of a set $S \subset X(K)$ such that $P \in \langle S \rangle$ or $+\infty$ if no such S exists, i.e. if $P \notin \langle X(K) \rangle$ ([3]). With this definition it is natural to analyze our proofs for an arbitrary field K.

Remark 1. Our proofs work verbatim if instead of $\mathbb{R}$ we take a real closed field K in the sense of [6], §1.2, and instead of $\mathbb{C}$ the algebraic closure $\overline{K}$ of K. We recall that a field K is real closed if and only if -1 is not a sum of squares of elements of K, each odd degree $f \in K[t]$ has a root in K and for each $a \in K$, either a or $-a$ has a square root in K. If K is a real closed field, then $\overline{K} \cong K[t]/(t^2 + 1)$ ([6], Theorem 1.2.2).

For curves our proofs give verbatim the following result.

Proposition 1. *Fix a field K such that* $char(K) = 0$ *and an integral and non-degenerate curve $X \subset \mathbb{P}^n$. Assume that both X and the embedding $X \hookrightarrow \mathbb{P}^n$ are defined over K and that $X(K)$ is infinite. Set $d := \deg(X)$. Assume that every $f \in K[t]$ of degree $d - n + 1$ has a root in K, i.e. assume the non-existence of a field extension $K \subset L$ such that* $\deg(L/K) = d - n + 1$. *Then $r_{X,K}(P) \leq n$ for all $P \in \mathbb{P}^n(K)$.*

The "i.e." part in Proposition 1 is true because every finite and separable extension of fields has a primitive element ([12], Theorem VII.5.4 on p. 156).

A small part of the inductive procedure in the proof of Theorem 1 works verbatim for an arbitrary field K such that $\mathrm{char}(K) = 0$. Indeed, for any $P \in \mathbb{P}^n(K)$ the set of A_P all hyperplanes of $\mathbb{P}^n(\overline{K})$ containing P is defined over K and $A_P(K)$ is dense in $A_P(\overline{K})$. However, the curve section C inductively obtained from X may have $C(K)$ finite. For instance, take as K a finite extension of $\mathbb{Q}$ and as X a smooth surface birational to $\mathbb{P}^2$ over K. The set $X(K)$ is Zariski dense in $X(\mathbb{C})$. Quite often, X has sectional genus at least 2. A theorem of Faltings (formerly Mordell's conjecture) says that $C(K)$ is finite for any integral curve C defined over K whose normalization has genus at least 2. We do not know a single field K (except the real closed ones and the algebraically closed ones) in which many curve sections C of a large class of varieties X have $C(K)$ infinite.

References

[1] Albera, L., Chevalier, P., Comon, P. and Ferreol, A., *On the virtual array concept for higher order array processing*, IEEE Trans. Signal Process. **53**(4) (2005), 1254–1271.

[2] Arbarello, E., Cornalba, M., Griffiths, P. and Harris, J., *Geometry of Algebraic Curves. I*, Springer-Verlag, New York, 1985.

[3] Ballico, E., *Ranks of subvarieties of $\mathbb{P}^n$ over non-algebraically closed fields*, Int. J. Pure Appl. Math. **61**(1) (2010), 7–10.

[4] Ballico, E., *Subsets of the variety $X \subset \mathbb{P}^n$ computing the X-rank of a point of $\mathbb{P}^n$*, preprint.

[5] Bernardi, A., Gimigliano, A. and Idà, M., *Computing symmetric rank for symmetric tensors*, J. Symbolic Comput. **46** (2011), 34–55.

[6] Bochnak, J., Coste, M. and Roy, F.-M., *Real Algebraic Geometry*, Translated from the 1987 French original. Revised by the authors. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 36, Springer-Verlag, Berlin, 1998.

[7] Buczyński, J., Landsberg, J. M., *Ranks of tensors and a generalization of secant varieties*, arXiv:0909.4262v1 [math.AG].

[8] Comas, G., Seiguer, M., *On the rank of a binary form*, arXiv:math.AG/0112311.

[9] Comon, P., Golub, G., Lim, L.-H. and Mourrain, B., *Symmetric tensors and symmetric tensor rank*, SIAM J. Matrix Anal. Appl. **30**(3) (2008), 1254–1279.

[10] Hartshorne, R., *Algebraic Geometry*, Springer-Verlag, Berlin, 1977.

[11] Landsberg, J. M., Teitler, Z., *On the ranks and border ranks of symmetric tensors*, Found. Comput. Math. **10** (2010), 339–366.

[12] Lang, S., *Algebra*, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965.

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Received January 8, 2010