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Jensen and Ostrowski type inequalities for general Lebesgue integral with applications

Abstract. Some inequalities related to Jensen and Ostrowski inequalities for general Lebesgue integral are obtained. Applications for f-divergence measure are provided as well.

1. Introduction. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu(t) = 1$. Consider the Lebesgue space

$$L(\Omega, \mu) := \left\{ f : \Omega \to \mathbb{R} \mid f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)| \, d\mu(t) < \infty \right\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t) \, d\mu(t)$.

The following reverse of the Jensen's inequality holds [12]:

Theorem 1. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, where $\mathring{I}$ is the interior of I. If $f : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \le f(t) \le M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

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and such that $f,\ \Phi\circ f\in L(\Omega,\mu)$, *then*

$$\begin{aligned} 0 &\leq \int_\Omega \Phi\circ f d\mu-\Phi\left(\int_\Omega f d\mu\right) \\ &\leq \left(M-\int_\Omega f d\mu\right)\left(\int_\Omega f d\mu - m\right)\frac{\Phi'_-(M)-\Phi'_+(m)}{M-m} \\ &\leq \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right], \end{aligned} \tag{1.1}$$

where Φ'_- *is the left and* Φ'_+ *is the right derivative of the convex function* Φ.

For other reverse of Jensen's inequality and applications to divergence measures see [12] and [15].

In 1938, A. Ostrowski [23] proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b \Phi(t)\,dt$ and the value $\Phi(x)$, $x\in[a,b]$.

Theorem 2. *Let* $\Phi:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *such that* $\Phi':(a,b)\to\mathbb{R}$ *is bounded on* (a,b), *i.e.,* $\|\Phi'\|_\infty := \sup_{t\in(a,b)}|\Phi'(t)|<\infty$. *Then*

$$\left|\Phi(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|\Phi'\|_\infty(b-a), \tag{1.2}$$

for all $x\in[a,b]$ *and the constant* $\frac{1}{4}$ *is the best possible.*

For various results related to Ostrowski's inequality see for instance [2], [3], [5]–[18] and the references therein.

Motivated by the above results, in this paper we investigate the magnitude of the quantity

$$\int_\Omega \Phi\circ g d\mu-\Phi(x)-\lambda\left(\int_\Omega g d\mu - x\right),\ x\in[a,b],$$

for various assumptions on the absolutely continuous function Φ, which in the particular case of $x=\int_\Omega g d\mu$ provides some results connected with Jensen's inequality while in the case $\lambda=0$ provides some generalizations of Ostrowski's inequality. Applications for divergence measures are provided as well.

2. Some identities.

The following result holds:

Lemma 1. *Let* $\Phi:I\to\mathbb{C}$ *be an absolutely continuous function on* $[a,b]\subset\mathring{I}$, *the interior of* I. *If* $g:\Omega\to[a,b]$ *is Lebesgue* μ*-measurable on* Ω *and such* *that* $\Phi\circ g,\ g\in L(\Omega,\mu)$, *then we have the equality*

$$\begin{aligned}&\int_\Omega \Phi\circ g d\mu-\Phi(x)-\lambda\left(\int_\Omega g d\mu-x\right)\\&=\int_\Omega\left[(g-x)\int_0^1\big(\Phi'((1-s)x+sg)-\lambda\big)ds\right]d\mu\end{aligned}\tag{2.1}$$

for any $\lambda\in\mathbb{C}$ *and* $x\in[a,b]$.

In particular, we have

$$\int_\Omega \Phi\circ g d\mu-\Phi(x)=\int_\Omega\left[(g-x)\int_0^1\Phi'((1-s)x+sg)\,ds\right]d\mu,\tag{2.2}$$

for any $x\in[a,b]$.

Proof. Since Φ is absolutely continuous on $[a,b]$, then for any $u,\ v\in[a,b]$ we have

$$\Phi(u)-\Phi(v)=(u-v)\int_0^1\Phi'((1-s)v+su)\,ds.\tag{2.3}$$

This implies that

$$\Phi(g(t))-\Phi(x)=(g(t)-x)\int_0^1\Phi'((1-s)x+sg(t))\,ds$$

for any $t\in\Omega$, or, equivalently

$$\Phi\circ g-\Phi(x)=(g-x)\int_0^1\Phi'((1-s)x+sg)\,ds.\tag{2.4}$$

Since $\Phi: I\to\mathbb{C}$ is an absolutely continuous functions on $[a,b]$, the Lebesgue integral over μ in the right side of (2.1) exists for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$.

Integrating (2.4) over the measure μ on Ω and since $\int_\Omega d\mu=1$, then we have

$$\int_\Omega \Phi\circ g d\mu-\Phi(x)=\int_\Omega\left[(g-x)\int_0^1\Phi'((1-s)x+sg)\,ds\right]d\mu.\tag{2.5}$$

Now, observe that for $\lambda\in\mathbb{C}$ we have

$$\begin{aligned}&\int_\Omega\left[(g-x)\int_0^1\big(\Phi'((1-s)x+sg)-\lambda\big)ds\right]d\mu\\&=\int_\Omega\left[(g-x)\left(\int_0^1\Phi'((1-s)x+sg)\,ds-\lambda\right)\right]d\mu\\&=\int_\Omega\left[(g-x)\int_0^1\Phi'((1-s)x+sg)\,ds\right]d\mu-\lambda\int_\Omega(g-x)\,d\mu\\&=\int_\Omega\left[(g-x)\int_0^1\Phi'((1-s)x+sg)\,ds\right]d\mu-\lambda\left(\int_\Omega g d\mu-x\right).\end{aligned}\tag{2.6}$$

Making use of (2.5) and (2.6), we deduce the desired result (2.1). □

Remark 1. With the assumptions of Lemma 1 we have

$$\begin{aligned}
&\int_{\Omega} \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) \\
&= \int_{\Omega} \left[\left(g - \frac{a+b}{2}\right) \int_0^1 \Phi'\left((1-s)\frac{a+b}{2} + sg\right) ds\right] d\mu.
\end{aligned} \tag{2.7}$$

Corollary 1. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}
&\int_{\Omega} \Phi \circ g d\mu - \Phi\left(\int_{\Omega} g d\mu\right) \\
&= \int_{\Omega} \left[\left(g - \int_{\Omega} g d\mu\right) \int_0^1 \Phi'\left((1-s)\int_{\Omega} g d\mu + sg\right) ds\right] d\mu.
\end{aligned} \tag{2.8}$$

Proof. We observe that since $g : \Omega \to [a, b]$ and $\int_{\Omega} d\mu = 1$, then $\int_{\Omega} g d\mu \in [a, b]$ and by taking $x = \int_{\Omega} g d\mu$ in (2.2), we get (2.8). □

Corollary 2. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}
&\int_{\Omega} \Phi \circ g d\mu - \frac{1}{b-a}\int_a^b \Phi(x)\, dx - \lambda\left(\int_{\Omega} g d\mu - \frac{a+b}{2}\right) \\
&= \int_{\Omega} \left\{\frac{1}{b-a}\int_a^b \left[(g-x)\int_0^1 \left(\Phi'((1-s)x + sg) - \lambda\right) ds\right] dx\right\} d\mu.
\end{aligned} \tag{2.9}$$

Proof. Follows by integrating the identity (2.1) over $x \in [a, b]$, dividing by $b - a > 0$ and using Fubini's theorem. □

Corollary 3. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous function on $[a, b] \subset \mathring{I}$, the interior of I. If $g, h : \Omega \to [a, b]$ are Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $\Phi \circ h$, g, $h \in L(\Omega, \mu)$, then we have the equality*

$$\begin{aligned}
&\int_{\Omega} \Phi \circ g d\mu - \int_{\Omega} \Phi \circ h d\mu - \lambda\left(\int_{\Omega} g d\mu - \int_{\Omega} h d\mu\right) \\
&= \int_{\Omega}\int_{\Omega} \left[(g(t) - h(\tau))\int_0^1 \left(\Phi'((1-s)h(\tau) + sg(t)) - \lambda\right) ds\right] \\
&\quad \times d\mu(t)\, d\mu(\tau)
\end{aligned} \tag{2.10}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

In particular, we have

$$\begin{aligned}
&\int_{\Omega} \Phi \circ g d\mu - \int_{\Omega} \Phi \circ h d\mu \\
&= \int_{\Omega}\int_{\Omega} \left[(g(t) - h(\tau))\int_0^1 \Phi'((1-s)h(\tau) + sg(t))\, ds\right] \\
&\quad \times d\mu(t)\, d\mu(\tau),
\end{aligned} \tag{2.11}$$

for any $x \in [a, b]$.

Proof. From (2.1) we have for any $\tau \in \Omega$ that

$$\begin{aligned}&\int_\Omega \Phi\circ g d\mu - \Phi(h(\tau)) - \lambda\left(\int_\Omega g d\mu - \Phi(h(\tau))\right)\\ &= \int_\Omega\left[(g-\Phi(h(\tau)))\int_0^1 \left(\Phi'((1-s)\Phi(h(\tau))+sg)-\lambda\right)ds\right]d\mu\end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$.

Integrating on Ω over $d\mu(\tau)$ and using Fubini's theorem, we get the desired result (2.10). □

Remark 2. The above equality (2.10) can be extended for two measures as follows

$$\begin{aligned}&\int_{\Omega_1}\Phi\circ g d\mu_1 - \int_{\Omega_2}\Phi\circ h d\mu_2 - \lambda\left(\int_{\Omega_1} g d\mu_1 - \int_{\Omega_2} h d\mu_2\right)\\ &= \int_{\Omega_1}\int_{\Omega_2}\left[(g(t)-h(\tau))\int_0^1\left(\Phi'((1-s)h(\tau)+sg(t))-\lambda\right)ds\right]\\ &\quad\times d\mu_1(t)\,d\mu_2(\tau),\end{aligned} \tag{2.12}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$ and provided that $\Phi\circ g$, $g \in L(\Omega_1,\mu_1)$ while $\Phi\circ h, h \in L(\Omega_2,\mu_2)$.

Remark 3. If $w \geq 0$ μ-almost everywhere (μ-a.e.) on Ω with $\int_\Omega w d\mu > 0$, then by replacing $d\mu$ with $\frac{w d\mu}{\int_\Omega w d\mu}$ in (2.1), we have the weighted equality

$$\begin{aligned}&\frac{1}{\int_\Omega w d\mu}\int_\Omega w(\Phi\circ g)d\mu - \Phi(x) - \lambda\left(\frac{1}{\int_\Omega w d\mu}\int_\Omega w g d\mu - x\right)\\ &= \frac{1}{\int_\Omega w d\mu}\int_\Omega w\cdot\left[(g-x)\int_0^1\left(\Phi'((1-s)x+sg)-\lambda\right)ds\right]d\mu\end{aligned} \tag{2.13}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$, provided $\Phi\circ g$, $g \in L_w(\Omega,\mu)$, where

$$L_w(\Omega,\mu) := \left\{g \middle|\ \int_\Omega w|g|d\mu < \infty\right\}.$$

The other equalities have similar weighted versions. However, the details are omitted.

If we use the discrete measure, then for a function $\Phi : I \to \mathbb{C}$ which is absolutely continuous on $[a,b] \subset \mathring{I}$, the interior of I, $x_j \in [a,b]$ and $p_j \geq 0$ with $\sum_{j=1}^n p_j = 1$, we can state the following identity

$$\begin{aligned}(2.14)\quad & \sum_{j=1}^{n} p_j \Phi(x_j) - \Phi(x) - \lambda\left(\sum_{j=1}^{n} p_j x_j - x\right) \\ & = \sum_{j=1}^{n} p_j \left[(x_j - x)\int_0^1 \big(\Phi'((1-s)x + s x_j) - \lambda\big)\, ds\right]\end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$.

In particular, we have

$$(2.15)\quad \sum_{j=1}^{n} p_j \Phi(x_j) - \Phi(x) = \sum_{j=1}^{n} p_j \left[(x_j - x)\int_0^1 \Phi'((1-s)x + s x_j)\, ds\right]$$

for any $x \in [a,b]$ and

$$\begin{aligned}(2.16)\quad & \sum_{j=1}^{n} p_j \Phi(x_j) - \Phi\left(\frac{a+b}{2}\right) \\ & = \sum_{j=1}^{n} p_j \left[\left(x_j - \frac{a+b}{2}\right)\int_0^1 \Phi'\left((1-s)\frac{a+b}{2} + s x_j\right) ds\right]\end{aligned}$$

and

$$\begin{aligned}(2.17)\quad & \sum_{j=1}^{n} p_j \Phi(x_j) - \Phi\left(\sum_{k=1}^{n} p_k x_k\right) \\ & = \sum_{j=1}^{n} p_j \left[\left(x_j - \sum_{k=1}^{n} p_k x_k\right)\int_0^1 \Phi'\left((1-s)\sum_{k=1}^{n} p_k x_k + s x_j\right) ds\right].\end{aligned}$$

If $x_j \in [a,b]$ and $p_j \geq 0$, $j \in \{1,\dots,n\}$ with $\sum_{j=1}^{n} p_j = 1$ and if $y_k \in [a,b]$ and $q_k \geq 0$, $k \in \{1,\dots,m\}$ with $\sum_{k=1}^{m} q_k = 1$, then we can state the following identity as well:

$$\begin{aligned}(2.18)\quad & \sum_{j=1}^{n} p_j \Phi(x_j) - \sum_{k=1}^{m} q_k \Phi(y_k) - \lambda\left(\sum_{j=1}^{n} p_j x_j - \sum_{k=1}^{m} q_k y_k\right) \\ & = \sum_{j=1}^{n} p_j \sum_{k=1}^{m} q_k \left[(x_j - y_k)\int_0^1 \big(\Phi'((1-s)y_k + s x_j) - \lambda\big)\, ds\right].\end{aligned}$$

In particular, we have

$$\begin{aligned}(2.19)\quad & \sum_{j=1}^{n} p_j \Phi(x_j) - \sum_{k=1}^{m} q_k \Phi(y_k) \\ & = \sum_{j=1}^{n} p_j \sum_{k=1}^{m} q_k \left[(x_j - y_k)\int_0^1 \Phi'((1-s)y_k + s x_j)\, ds\right].\end{aligned}$$

3. Bounds in terms of p-norms. We use the notations

$$\|k\|_{\Omega,p} := \begin{cases} \left(\int\limits_{\Omega} |k(t)|^p \, d\mu(t)\right)^{1/p} < \infty, & p \geq 1, \ k \in L_p(\Omega,\mu); \\ \operatorname{ess\,sup}_{t\in\Omega} |k(t)| < \infty, & p = \infty, \ k \in L_\infty(\Omega,\mu); \end{cases}$$

and

$$\|\Phi\|_{[0,1],p} := \begin{cases} \left(\int_0^1 |\Phi(s)|^p \, ds\right)^{1/p} < \infty, & p \geq 1, \ \Phi \in L_p(0,1); \\ \operatorname{ess\,sup}_{s\in[0,1]} |\Phi(s)| < \infty, & p = \infty, \ \Phi \in L_\infty(0,1). \end{cases}$$

If we consider the identity function $\ell : [0,1] \to [0,1]$, $\ell(s) = s$ we have

$$\int_0^1 \left|\Phi'((1-s)x + sg(t)) - \lambda\right|^p ds = \left\|\Phi'((1-\ell)x + \ell g(t)) - \lambda\right\|_{[0,1],p}^p$$

and

$$\operatorname*{ess\,sup}_{s\in[0,1]} \left|\Phi'((1-s)x + sg(t)) - \lambda\right| = \left\|\Phi'((1-\ell)x + \ell g(t)) - \lambda\right\|_{[0,1],\infty}$$

for $t \in \Omega$.

Theorem 3. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous function on $[a,b] \subset \mathring{I}$, the interior of I. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega,\mu)$, then*

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \lambda\left(\int_\Omega g d\mu - x\right)\right| \\ &\leq \int_\Omega |g-x| \left\|\Phi'((1-\ell)x + \ell g) - \lambda\right\|_{[0,1],1} d\mu \\ &\leq \begin{cases} \|g-x\|_{\Omega,\infty} \left\|\|\Phi'((1-\ell)x + \ell g) - \lambda\|_{[0,1],1}\right\|_{\Omega,1}; \\ \|g-x\|_{\Omega,p} \left\|\|\Phi'((1-\ell)x + \ell g) - \lambda\|_{[0,1],1}\right\|_{\Omega,q} \\ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \|g-x\|_{\Omega,1} \left\|\|\Phi'((1-\ell)x + \ell g) - \lambda\|_{[0,1],1}\right\|_{\Omega,\infty}; \end{cases} \end{aligned} \tag{3.1}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$. In particular, we have

$$\left|\int_\Omega \Phi\circ g d\mu-\Phi(x)\right| \tag{3.2}$$
$$\le \int_\Omega |g-x| \left\|\Phi'((1-\ell)x+\ell g)\right\|_{[0,1],1} d\mu$$
$$\le \begin{cases} \|g-x\|_{\Omega,\infty}\left\|\|\Phi'((1-\ell)x+\ell g)\|_{[0,1],1}\right\|_{\Omega,1}; \\ \|g-x\|_{\Omega,p}\left\|\|\Phi'((1-\ell)x+\ell g)\|_{[0,1],1}\right\|_{\Omega,q} \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|g-x\|_{\Omega,1}\left\|\|\Phi'((1-\ell)x+\ell g)\|_{[0,1],1}\right\|_{\Omega,\infty}; \end{cases}$$

for any $x\in[a,b]$.

Proof. Taking the modulus in the equality (2.1), we have

$$\left|\int_\Omega \Phi\circ g d\mu-\Phi(x)-\lambda\left(\int_\Omega g d\mu-x\right)\right| \tag{3.3}$$
$$\le \int_\Omega \left|(g-x)\int_0^1 \left(\Phi'((1-s)x+sg)-\lambda\right) ds\right| d\mu$$
$$\le \int_\Omega |g-x|\int_0^1 \left|\Phi'((1-s)x+sg)-\lambda\right| ds d\mu$$
$$= \int_\Omega |g-x| \left\|\Phi'((1-\ell)x+\ell g)-\lambda\right\|_{[0,1],1} d\mu$$

for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$.

Utilising Hölder's inequality for the μ-measurable functions $F,G:\Omega\to\mathbb{C}$,

$$\left|\int_\Omega FG d\mu\right| \le \left(\int_\Omega |F|^p d\mu\right)^{1/p}\left(\int_\Omega |G|^q d\mu\right)^{1/q},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1$$

and

$$\left|\int_\Omega FG d\mu\right| \le \operatorname*{ess\,sup}_{t\in\Omega}|F(t)|\int_\Omega |G| d\mu$$

we get from (3.3) the desired result (3.1). □

Remark 4. If we take $x=\frac{a+b}{2}$ in (3.1), then we get

$$\left|\int_\Omega \Phi\circ g d\mu-\Phi\left(\frac{a+b}{2}\right)-\lambda\left(\int_\Omega g d\mu-\frac{a+b}{2}\right)\right| \tag{3.4}$$
$$\le \int_\Omega \left|g-\frac{a+b}{2}\right| \left\|\Phi'\left((1-\ell)\frac{a+b}{2}+\ell g\right)-\lambda\right\|_{[0,1],1} d\mu$$

$$\leq \begin{cases} \left\|g-\frac{a+b}{2}\right\|_{\Omega,\infty}\left\|\left\|\Phi'\left((1-\ell)\frac{a+b}{2}+\ell g\right)-\lambda\right\|_{[0,1],1}\right\|_{\Omega,1}; \\ \left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left\|\left\|\Phi'\left((1-\ell)\frac{a+b}{2}+\ell g\right)-\lambda\right\|_{[0,1],1}\right\|_{\Omega,q} \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left\|g-\frac{a+b}{2}\right\|_{\Omega,1}\left\|\left\|\Phi'\left((1-\ell)\frac{a+b}{2}+\ell g\right)-\lambda\right\|_{[0,1],1}\right\|_{\Omega,\infty}; \end{cases}$$

for any $\lambda \in \mathbb{C}$ and, in particular, for $\lambda = 0$ we have

$$\begin{aligned} &\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\frac{a+b}{2}\right)\right| \\ &\quad\leq \int_\Omega \left|g-\frac{a+b}{2}\right|\left\|\Phi'\left((1-\ell)\frac{a+b}{2}+\ell g\right)\right\|_{[0,1],1} d\mu \\ &\quad\leq \begin{cases} \left\|g-\frac{a+b}{2}\right\|_{\Omega,\infty}\left\|\left\|\Phi'\left((1-\ell)\frac{a+b}{2}+\ell g\right)\right\|_{[0,1],1}\right\|_{\Omega,1}; \\ \left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left\|\left\|\Phi'\left((1-\ell)\frac{a+b}{2}+\ell g\right)\right\|_{[0,1],1}\right\|_{\Omega,q} \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left\|g-\frac{a+b}{2}\right\|_{\Omega,1}\left\|\left\|\Phi'\left((1-\ell)\frac{a+b}{2}+\ell g\right)\right\|_{[0,1],1}\right\|_{\Omega,\infty}; \end{cases} \end{aligned} \tag{3.5}$$

If we take $x = \int_\Omega g d\mu$ in (3.1), then we get

$$\begin{aligned} &\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \\ &\leq \int_\Omega \left|g-\int_\Omega g d\mu\right|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu+\ell g\right)-\lambda\right\|_{[0,1],1} d\mu \\ &\leq \begin{cases} \left\|g-\int_\Omega g d\mu\right\|_{\Omega,\infty}\left\|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu+\ell g\right)-\lambda\right\|_{[0,1],1}\right\|_{\Omega,1}; \\ \left\|g-\int_\Omega g d\mu\right\|_{\Omega,p}\left\|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu+\ell g\right)-\lambda\right\|_{[0,1],1}\right\|_{\Omega,q} \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left\|g-\int_\Omega g d\mu\right\|_{\Omega,1}\left\|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu+\ell g\right)-\lambda\right\|_{[0,1],1}\right\|_{\Omega,\infty}; \end{cases} \end{aligned} \tag{3.6}$$

for any $\lambda \in \mathbb{C}$ and, in particular, for $\lambda = 0$ we have

$$
\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \\
&\le \int_\Omega \left|g-\int_\Omega g d\mu\right| \left\|\Phi'\left((1-\ell)\int_\Omega g d\mu + \ell g\right)\right\|_{[0,1],1} d\mu \\
&\le \begin{cases} \left\|g-\int_\Omega g d\mu\right\|_{\Omega,\infty} \left\|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu + \ell g\right)\right\|_{[0,1],1}\right\|_{\Omega,1}; \\ \left\|g-\int_\Omega g d\mu\right\|_{\Omega,p} \left\|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu + \ell g\right)\right\|_{[0,1],1}\right\|_{\Omega,q} \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left\|g-\int_\Omega g d\mu\right\|_{\Omega,1} \left\|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu + \ell g\right)\right\|_{[0,1],1}\right\|_{\Omega,\infty}. \end{cases}
\end{aligned} \tag{3.7}
$$

Corollary 4. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous function on $[a,b] \subset \mathring{I}$, the interior of I. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g,\ g \in L(\Omega,\mu)$, then*

$$\left|\int_\Omega \Phi\circ g d\mu - \Phi(x)\right| \le \left\|\Phi'\right\|_{[a,b],\infty} \int_\Omega |g-x|\, d\mu \tag{3.8}$$

for any $x\in[a,b]$.

In particular, we have

$$\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\frac{a+b}{2}\right)\right| \le \left\|\Phi'\right\|_{[a,b],\infty} \int_\Omega \left|g-\frac{a+b}{2}\right| d\mu \tag{3.9}$$

and

$$\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \le \left\|\Phi'\right\|_{[a,b],\infty} \int_\Omega \left|g-\int_\Omega g d\mu\right| d\mu. \tag{3.10}$$

Proof. We have from (3.1) that

$$\left|\int_\Omega \Phi\circ g d\mu - \Phi(x)\right| \le \int_\Omega |g-x| \left(\int_0^1 \left|\Phi'((1-s)x+sg)\right| ds\right) d\mu \tag{3.11}$$

for any $x\in[a,b]$.

However, for any $t\in\Omega$ and almost every $s\in[0,1]$ we have

$$\left|\Phi'((1-s)x+sg(t))\right| \le \operatorname*{ess\,sup}_{u\in[a,b]} \left|\Phi'(u)\right| = \left\|\Phi'\right\|_{[a,b],\infty},$$

for any $x\in[a,b]$.

Making use of (3.11), we get (3.8). □

Remark 5. We remark that the quantity from Corollary 4

$$\delta_\mu(g,x) := \int_\Omega |g-x|\, d\mu$$

cannot be computed in general.

However, in the case when $\Omega = [a,b]$, $g : [a,b] \to [a,b]$, $g(t) = t$ and $\mu(t) = \frac{1}{b-a} dt$, we have

$$\begin{aligned}\delta_\mu(g,x) &:= \frac{1}{b-a}\int_a^b |t-x|\,dt = \frac{1}{b-a}\left[\int_a^x (x-t)\,dt + \int_x^b (t-x)\,dt\right] \\ &= \frac{1}{b-a}\left[(x-a)^2 + (b-x)^2\right] \\ &= \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a),\end{aligned}$$

where $x \in [a,b]$.

Utilising the inequality (3.8), we get Ostrowski's inequality

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi(x)\right| \\ &\le \left\|\Phi'\right\|_{[a,b],\infty}\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\end{aligned} \tag{3.12}$$

for any $x \in [a,b]$.

From the inequalities (3.9) and (3.10) we get the midpoint inequality

$$\left|\frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi\left(\frac{a+b}{2}\right)\right| \le \frac{1}{4}\left\|\Phi'\right\|_{[a,b],\infty}(b-a). \tag{3.13}$$

Remark 6. If we consider the dispersion or the standard variation

$$\sigma_\mu(g) := \left(\int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu\right)^{1/2} = \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2},$$

then by (3.10) we have the inequalities

$$\begin{aligned}\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| &\le \left\|\Phi'\right\|_{[a,b],\infty}\delta_\mu\left(g, \int_\Omega g d\mu\right) \\ &\le \left\|\Phi'\right\|_{[a,b],\infty}\sigma_\mu(g).\end{aligned} \tag{3.14}$$

In general, we have by Cauchy–Bunyakovsky–Schwarz's inequality that

$$\delta_\mu(g,x) := \int_\Omega |g-x|\,d\mu \le \left(\int_\Omega (g-x)^2\,d\mu\right)^{1/2}. \tag{3.15}$$

Since

$$\begin{aligned}&\int_\Omega (g-x)^2\,d\mu = \int_\Omega \left(g - \int_\Omega g d\mu + \int_\Omega g d\mu - x\right)^2 d\mu \\ &= \int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu + 2\int_\Omega \left(g - \int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right) d\mu\end{aligned}$$

$$+\int_{\Omega}\left(\int_{\Omega}gd\mu-x\right)^{2}d\mu$$
$$=\int_{\Omega}\left(g-\int_{\Omega}gd\mu\right)^{2}d\mu+\left(\int_{\Omega}gd\mu-x\right)^{2}$$

for any $x\in[a,b]$, then by (3.8) and (3.15) we get the inequalities

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi(x)\right|\leq\left\|\Phi'\right\|_{[a,b],\infty}\delta_{\mu}(g,x)\leq\left\|\Phi'\right\|_{[a,b],\infty}\left[\sigma_{\mu}^{2}(g)+\left(\int_{\Omega}gd\mu-x\right)^{2}\right]^{1/2}\tag{3.16}$$

for any $x\in[a,b]$.

If we use the discrete measure, then from (3.16) we have

$$\begin{aligned}&\left|\sum_{j=1}^{n}p_{j}\Phi(x_{j})-\Phi(x)\right|\\&\leq\left\|\Phi'\right\|_{[a,b],\infty}\sum_{j=1}^{n}p_{j}\left|x_{j}-x\right|\\&\leq\left\|\Phi'\right\|_{[a,b],\infty}\left[\sum_{j=1}^{n}p_{j}x_{j}^{2}-\left(\sum_{j=1}^{n}p_{j}x_{j}\right)^{2}+\left(\sum_{j=1}^{n}p_{j}x_{j}-x\right)^{2}\right]^{1/2},\end{aligned}\tag{3.17}$$

for any $x\in[a,b]$, where $x_j\in[a,b]$ and $p_j\geq 0$ with $\sum_{j=1}^{n}p_j=1$.

In particular, we have

$$\left|\sum_{j=1}^{n}p_{j}\Phi(x_{j})-\Phi\left(\frac{a+b}{2}\right)\right|\leq\left\|\Phi'\right\|_{[a,b],\infty}\sum_{j=1}^{n}p_{j}\left|x_{j}-\frac{a+b}{2}\right|\leq\frac{1}{2}(b-a)\left\|\Phi'\right\|_{[a,b],\infty}\tag{3.18}$$

and

$$\begin{aligned}&\left|\sum_{j=1}^{n}p_{j}\Phi(x_{j})-\Phi\left(\sum_{k=1}^{n}p_{k}x_{k}\right)\right|\leq\left\|\Phi'\right\|_{[a,b],\infty}\sum_{j=1}^{n}p_{j}\left|x_{j}-\sum_{k=1}^{n}p_{k}x_{k}\right|\\&\leq\left\|\Phi'\right\|_{[a,b],\infty}\left[\sum_{j=1}^{n}p_{j}x_{j}^{2}-\left(\sum_{j=1}^{n}p_{j}x_{j}\right)^{2}\right]^{1/2}\\&\leq\frac{1}{2}(b-a)\left\|\Phi'\right\|_{[a,b],\infty}.\end{aligned}\tag{3.19}$$

4. Inequalities for bounded derivatives. Now, for γ, $\Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} | \operatorname{Re}\left[(\Gamma - f(t)) \left(\overline{f(t)} - \overline{\gamma} \right) \right] \geq 0 \text{ for a.e. } t \in [a,b] \right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} | \left| f(t) - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2}|\Gamma - \gamma| \text{ for a.e. } t \in [a,b] \right\}.$$

The following representation result may be stated.

Proposition 1. *For any γ, $\Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{4.1}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2}|\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma}) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma - \gamma|^2 - \left| z - \frac{\gamma+\Gamma}{2} \right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma}) \right]$$

that holds for any $z \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties, we can also state that:

Corollary 5. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have*

$$\begin{aligned} &\bar{U}_{[a,b]}(\gamma,\Gamma) \\ &= \{ f : [a,b] \to \mathbb{C} \mid (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) \\ &\quad + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \geq 0 \text{ for a.e. } t \in [a,b] \}. \end{aligned} \tag{4.2}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\gamma,\Gamma) := \{ f : [a,b] \to \mathbb{C} \mid & \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \\ & \text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for a.e. } t \in [a,b] \}. \end{aligned} \tag{4.3}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{4.4}$$

The following result holds:

Theorem 4. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous function on $[a,b] \subset \mathring{I}$, the interior of I. For some $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, assume that $\Phi' \in \bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega,\mu)$, then we have the inequality*

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - x\right)\right| \\ &\leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega |g-x| \, d\mu \end{aligned} \tag{4.5}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \\ &\leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu \leq \frac{1}{4}(b-a)|\Gamma-\gamma| \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| &\leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega \left|g - \int_\Omega g d\mu\right| d\mu \\ &\leq \frac{1}{2}|\Gamma-\gamma| \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)|\Gamma-\gamma|. \end{aligned} \tag{4.7}$$

Proof. By the equality (2.1) for $\lambda = \frac{\gamma+\Gamma}{2}$ we have

$$\begin{aligned} &\int_\Omega \Phi \circ g d\mu - \Phi(x) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - x\right) \\ &= \int_\Omega \left[(g-x)\int_0^1 \left(\Phi'((1-s)x+sg) - \frac{\gamma+\Gamma}{2}\right) ds\right] d\mu. \end{aligned} \tag{4.8}$$

Since $\Phi' \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, we have

$$\left|\Phi'((1-s)x+sg(t)) - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma-\gamma| \tag{4.9}$$

for a.e. $s \in [0,1]$ and for any $x \in [a,b]$ and any $t \in \Omega$.

Integrating (4.9) over s on $[0,1]$, we get

$$\int_0^1 \left|\Phi'((1-s)x+sg(t)) - \frac{\gamma+\Gamma}{2}\right| ds \leq \frac{1}{2}|\Gamma-\gamma| \tag{4.10}$$

for any $x \in [a,b]$ and any $t \in \Omega$.

Taking the modulus in (4.8), we get via (4.10) that

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu-\Phi(x)-\frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - x\right)\right| \\
&\le \int_\Omega\left[|g-x|\left|\int_0^1\left(\Phi'((1-s)x+sg)-\frac{\gamma+\Gamma}{2}\right)ds\right|\right]d\mu \\
&\le \int_\Omega\left[|g-x|\int_0^1\left|\Phi'((1-s)x+sg(t))-\frac{\gamma+\Gamma}{2}\right|ds\right]d\mu \\
&\le \frac{1}{2}|\Gamma-\gamma|\int_\Omega |g-x|\,d\mu
\end{aligned} \tag{4.11}$$

and the proof of (4.5) is completed. □

Corollary 6. *Let $\Phi : I \to \mathbb{R}$ be a convex function on $[a,b]\subset \mathring{I}$, the interior of I. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g$, $g\in L(\Omega,\mu)$, then we have the inequality*

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu-\Phi(x)-\frac{\Phi'_+(a)+\Phi'_-(b)}{2}\left(\int_\Omega g d\mu - x\right)\right| \\
&\le \frac{1}{2}\left[\Phi'_-(b)-\Phi'_+(a)\right]\int_\Omega |g-x|\,d\mu
\end{aligned} \tag{4.12}$$

for any $x\in[a,b]$.

In particular, we have

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu-\Phi\left(\frac{a+b}{2}\right)-\frac{\Phi'_+(a)+\Phi'_-(b)}{2}\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \\
&\le \frac{1}{2}\left[\Phi'_-(b)-\Phi'_+(a)\right]\int_\Omega \left|g-\frac{a+b}{2}\right|d\mu \\
&\le \frac{1}{4}(b-a)\left[\Phi'_-(b)-\Phi'_+(a)\right]
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
0 &\le \int_\Omega \Phi\circ g d\mu-\Phi\left(\int_\Omega g d\mu\right) \\
&\le \frac{1}{2}\left[\Phi'_-(b)-\Phi'_+(a)\right]\int_\Omega \left|g-\int_\Omega g d\mu\right|d\mu.
\end{aligned} \tag{4.14}$$

The discrete case is as follows:

Remark 7. Let $\Phi : I\to\mathbb{C}$ be an absolutely continuous function on $[a,b]\subset \mathring{I}$, the interior of I. For some $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, assume that $\Phi'\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$.

If $x_j \in [a,b]$ and $p_j \geq 0$ with $\sum_{j=1}^{n} p_j = 1$ then we have the inequality

$$\left| \sum_{j=1}^{n} p_j \Phi(x_j) - \Phi(x) - \frac{\gamma + \Gamma}{2} \left(\sum_{k=1}^{n} p_k x_k - x \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \sum_{j=1}^{n} p_j |x_j - x| \tag{4.15}$$

for any $x \in [a,b]$.

In particular, we have

$$\left| \sum_{j=1}^{n} p_j \Phi(x_j) - \Phi\left(\frac{a+b}{2}\right) - \frac{\gamma + \Gamma}{2} \left(\sum_{k=1}^{n} p_k x_k - \frac{a+b}{2} \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \sum_{j=1}^{n} p_j \left| x_j - \frac{a+b}{2} \right| \leq \frac{1}{4} (b-a) |\Gamma - \gamma| \tag{4.16}$$

and

$$\left| \sum_{j=1}^{n} p_j \Phi(x_j) - \Phi\left(\sum_{k=1}^{n} p_k x_k \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \sum_{j=1}^{n} p_j \left| x_j - \sum_{k=1}^{n} p_k x_k \right|$$
$$\leq \frac{1}{2} |\Gamma - \gamma| \left(\sum_{j=1}^{n} p_j x_j^2 - \left(\sum_{k=1}^{n} p_k x_k \right)^2 \right)^{1/2} \leq \frac{1}{4} (b-a) |\Gamma - \gamma|.$$

If $\Phi : I \to \mathbb{R}$ is a convex function on $[a,b]$, then we have

$$\left| \sum_{j=1}^{n} p_j \Phi(x_j) - \Phi(x) - \frac{\Phi'_+(a) + \Phi'_-(b)}{2} \left(\sum_{k=1}^{n} p_k x_k - x \right) \right| \leq \frac{1}{2} \left[\Phi'_-(b) - \Phi'_+(a) \right] \sum_{j=1}^{n} p_j |x_j - x|$$

for any $x \in [a,b]$.

In particular, we have

$$\left| \sum_{j=1}^{n} p_j \Phi(x_j) - \Phi\left(\frac{a+b}{2}\right) - \frac{\Phi'_+(a) + \Phi'_-(b)}{2} \left(\sum_{k=1}^{n} p_k x_k - \frac{a+b}{2} \right) \right|$$
$$\leq \frac{1}{2} \left[\Phi'_-(b) - \Phi'_+(a) \right] \sum_{j=1}^{n} p_j \left| x_j - \frac{a+b}{2} \right|$$
$$\leq \frac{1}{4} (b-a) \left[\Phi'_-(b) - \Phi'_+(a) \right].$$

5. Applications for f-divergence. Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be $\mathcal{P} := \left\{p \mid p : \Omega \to \mathbb{R},\, p(t) \geq 0,\, \int_\Omega p(t)\, d\mu(t) = 1\right\}$. The Kullback–Leibler divergence [22] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_\Omega p(t) \ln\left[\frac{p(t)}{q(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{5.1}$$

where ln is to base e.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback–Leibler divergence. These are the: *variation distance* D_v, *Hellinger distance* D_H [19], χ^2*-divergence* D_{χ^2}, α*-divergence* D_α, *Bhattacharyya distance* D_B [1], *harmonic distance* D_{Ha}, *Jeffrey's distance* D_J [20], *triangular discrimination* D_Δ [25], etc. They are defined as follows:

$$D_v(p,q) := \int_\Omega |p(t) - q(t)|\, d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.2}$$

$$D_H(p,q) := \int_\Omega \left|\sqrt{p(t)} - \sqrt{q(t)}\right| d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.3}$$

$$D_{\chi^2}(p,q) := \int_\Omega p(t) \left[\left(\frac{q(t)}{p(t)}\right)^2 - 1\right] d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.4}$$

$$D_\alpha(p,q) := \frac{4}{1-\alpha^2}\left[1 - \int_\Omega [p(t)]^{\frac{1-\alpha}{2}} [q(t)]^{\frac{1+\alpha}{2}} d\mu(t)\right], \quad p,q \in \mathcal{P}; \tag{5.5}$$

$$D_B(p,q) := \int_\Omega \sqrt{p(t)\, q(t)}\, d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.6}$$

$$D_{Ha}(p,q) := \int_\Omega \frac{2p(t)\, q(t)}{p(t) + q(t)} d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.7}$$

$$D_J(p,q) := \int_\Omega [p(t) - q(t)] \ln\left[\frac{p(t)}{q(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.8}$$

$$D_\Delta(p,q) := \int_\Omega \frac{[p(t) - q(t)]^2}{p(t) + q(t)} d\mu(t), \quad p,q \in \mathcal{P}. \tag{5.9}$$

For other divergence measures, see the paper [21] by Kapur or the book online [24] by Taneja.

Csiszár f-divergence is defined as follows [4]

$$I_f(p,q) := \int_\Omega p(t) f\left[\frac{q(t)}{p(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{5.10}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u=1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)–(5.9) are particular instances of Csiszár f-divergence. There are also many others which are not in this class (see for example [24]).

The following result holds:

Proposition 2. *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function with the property that $f(1)=0$. Assume that $p,\ q\in\mathcal{P}$ and there exist constants $0<r<1<R<\infty$ such that*

$$r\le\frac{q(t)}{p(t)}\le R\ \ \text{for } \mu\text{-a.e. } t\in\Omega. \tag{5.11}$$

If $x\in[r,R]$, then we have the inequalities

$$\begin{aligned}|I_f(p,q)-f(x)|&\le\left\|f'\right\|_{[r,R],\infty}D_{v,x}(p,q)\\&\le\left\|f'\right\|_{[r,R],\infty}\left[D_{\chi^2}(p,q)+(x-1)^2\right]^{1/2},\end{aligned} \tag{5.12}$$

where

$$D_{v,x}(p,q):=\int_\Omega|p(t)-xq(t)|\,d\mu(t),\ \ p,q\in\mathcal{P}.$$

In particular, we have

$$\begin{aligned}\left|I_f(p,q)-f\left(\frac{r+R}{2}\right)\right|&\le\left\|f'\right\|_{[r,R],\infty}D_{v,\frac{r+R}{2}}(p,q)\\&\le\left\|f'\right\|_{[r,R],\infty}\left[D_{\chi^2}(p,q)+\left(\frac{r+R}{2}-1\right)^2\right]^{1/2}\end{aligned} \tag{5.13}$$

and

$$0\le I_f(p,q)\le\left\|f'\right\|_{[r,R],\infty}D_v(p,q)\le\left\|f'\right\|_{[r,R],\infty}\left[D_{\chi^2}(p,q)\right]^{1/2}. \tag{5.14}$$

Proof. Utilising the inequality (3.16) for the convex function f, we have

$$\begin{aligned}&\left|\int_\Omega p(t)f\left[\frac{q(t)}{p(t)}\right]d\mu(t)-f(x)\right|\\&\le\left\|f'\right\|_{[r,R],\infty}\int_\Omega\left|\frac{q(t)}{p(t)}-x\right|p(t)\,d\mu(t)\\&\le\left\|f'\right\|_{[r,R],\infty}\left[\int_\Omega\left(\frac{q(t)}{p(t)}\right)^2p(t)\,d\mu(t)-\left(\int_\Omega\frac{q(t)}{p(t)}p(t)\,d\mu(t)\right)^2\right.\\&\qquad\left.+\left(\int_\Omega\frac{q(t)}{p(t)}p(t)\,d\mu(t)-x\right)^2\right]^{1/2},\end{aligned}$$

which is equivalent to (5.12). □

We also have

Proposition 3. *With the assumptions of Proposition 2 we have*

$$\left|I_f(p,q) - f(x) - \frac{f'_+(r)+f'_-(R)}{2}(1-x)\right| \le \frac{1}{2}\left[f'_-(R)-f'_+(r)\right] D_{v,x}(p,q) \tag{5.15}$$

for any $x \in [r,R]$.

In particular, we have

$$\left|I_f(p,q) - f\left(\frac{r+R}{2}\right) - \frac{f'_+(r)+f'_-(R)}{2}\left(1-\frac{r+R}{2}\right)\right| \le \frac{1}{2}\left[f'_-(R)-f'_+(r)\right] D_{v,\frac{r+R}{2}}(p,q) \tag{5.16}$$

and

$$0 \le I_f(p,q) \le \frac{1}{2}\left[f'_-(R)-f'_+(r)\right] D_v(p,q). \tag{5.17}$$

Proof. Utilising the inequality (4.12), we have

$$\begin{aligned}&\left|\int_\Omega p(t) f\left[\frac{q(t)}{p(t)}\right] d\mu(t) - f(x) - \frac{f'_+(r)+f'_-(R)}{2}\left(\int_\Omega \frac{q(t)}{p(t)} p(t)\, d\mu(t) - x\right)\right| \\ &\le \frac{1}{2}\left[f'_-(R)-f'_+(r)\right]\int_\Omega \left|\frac{q(t)}{p(t)} - x\right| p(t)\, d\mu(t)\end{aligned}$$

for any $x \in [a,b]$, which is equivalent to (5.15). □

If we consider the convex function $f:(0,\infty)\to\mathbb{R}$, $f(t) = t\ln t$, then

$$\begin{aligned}I_f(p,q) &:= \int_\Omega p(t)\frac{q(t)}{p(t)}\ln\left[\frac{q(t)}{p(t)}\right] d\mu(t) = \int_\Omega q(t)\ln\left[\frac{q(t)}{p(t)}\right] d\mu(t) \\ &= D_{KL}(q,p).\end{aligned}$$

We have $f'(t) = \ln t + 1$ and by (5.17) we get

$$0 \le D_{KL}(q,p) \le \ln\sqrt{\frac{R}{r}} D_v(p,q), \tag{5.18}$$

provided that

$$r \le \frac{q(t)}{p(t)} \le R \text{ for } \mu\text{-a.e. } t \in \Omega.$$

If we consider the convex function $f:(0,\infty)\to\mathbb{R}$, $f(t) = -\ln t$, then

$$\begin{aligned}I_f(p,q) &:= -\int_\Omega p(t)\ln\left[\frac{q(t)}{p(t)}\right] d\mu(t) = \int_\Omega p(t)\ln\left[\frac{p(t)}{q(t)}\right] d\mu(t) \\ &= D_{KL}(p,q).\end{aligned}$$

We have $f'(t) = -\frac{1}{t}$ and by (5.17) we get

$$0 \le D_{KL}(p,q) \le \frac{1}{2}\frac{R-r}{rR}D_v(p,q) \tag{5.19}$$

provided that

$$r \le \frac{q(t)}{p(t)} \le R \text{ for } \mu\text{-a.e. } t \in \Omega.$$

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