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Entire functions of exponential type not vanishing in the half-plane $\Im z > k$, where $k > 0$

ABSTRACT. Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k$, $k \le 1$, and let $Q(z) := z^n\overline{P(1/\bar{z})}$. It was shown by Govil that if $\max_{|z|=1}|P'(z)|$ and $\max_{|z|=1}|Q'(z)|$ are attained at the same point of the unit circle $|z| = 1$, then

$$\max_{|z|=1}|P'(z)| \le \frac{n}{1+k^n}\max_{|z|=1}|P(z)|.$$

The main result of the present article is a generalization of Govil's polynomial inequality to a class of entire functions of exponential type.

1. Introduction and statement of results.

1.1. Bernstein's inequality for trigonometric polynomials. Let $\mathcal{P}_m$ denote the class of all polynomials of degree at most m and let $Q(z) := z^m\overline{P\left(1/\bar{z}\right)}$. It is well known that if $P \in \mathcal{P}_n$ and $|P(z)| \le M$ for $|z| = 1$, then (see [9, p. 524])

$$\left|Q'(z)\right| + \left|P'(z)\right| \le Mn \qquad (|z| = 1).$$

This result includes Bernstein's inequality for polynomials

$$\left|P'(z)\right| \le Mn \qquad (|z| = 1), \tag{1.1}$$

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and yields to

$$\left|\frac{\mathrm{d}}{\mathrm{d}\theta}P(\mathrm{e}^{\mathrm{i}\theta})\right| + \left|-\mathrm{i}nP(\mathrm{e}^{\mathrm{i}\theta}) + \frac{\mathrm{d}}{\mathrm{d}\theta}P(\mathrm{e}^{\mathrm{i}\theta})\right| \le Mn \qquad (\theta \in \mathbb{R})\,.$$

If $t(\theta) := \sum_{\nu=-n}^{n} a_\nu \mathrm{e}^{\mathrm{i}\nu\theta}$ is a trigonometric polynomial and $|t(\theta)| \le M$ for all real $\theta \in \mathbb{R}$, then $\mathrm{e}^{\mathrm{i}n\theta}t(\theta) = P(\mathrm{e}^{\mathrm{i}\theta})$, where $P \in \mathcal{P}_{2n}$ and $|P(z)| \le M$ for $|z| = 1$. Applying the preceding inequality with $2n$ instead of n, we obtain

$$|\mathrm{i}nt(\theta) + t'(\theta)| + |-\mathrm{i}nt(\theta) + t'(\theta)| \le 2Mn \qquad (\theta \in \mathbb{R})\,. \tag{1.2}$$

In particular, we have

$$|2t'(\theta)| \le |\mathrm{i}nt(\theta) + t'(\theta)| + |-\mathrm{i}nt(\theta) + t'(\theta)| \le 2Mn \qquad (\theta \in \mathbb{R})\,,$$

that is,

$$|t'(\theta)| \le Mn \qquad (\theta \in \mathbb{R})\,. \tag{1.3}$$

This is the famous inequality of S. Bernstein for trigonometric polynomials. It is sharp and in (1.2), equality can hold at any point $\theta \in \mathbb{R}$.

From (1.2) it follows that *if* $t(\theta)$ *is real for all real* θ*, then*

$$n^2t^2(\theta) + \big(t'(\theta)\big)^2 \le M^2n^2 \qquad (\theta \in \mathbb{R})\,. \tag{1.4}$$

Remark. Bernstein had proved (1.3) for cosine polynomials and also for sine polynomials. M. Riesz [10] seems to have been the first to prove it in its full generality. Inequality (1.4) is a result of J. G. van der Corput and G. Shaake [5].

2. Functions of exponential and Bernstein's inequality.

Basic properties of functions of exponential type. A trigonometric polynomial $t(\theta) := \sum_{\nu=-n}^{n} a_\nu \mathrm{e}^{\mathrm{i}\nu\theta}$ is well defined for any θ in the complex plane $\mathbb{C}$ and not only when θ is restricted to the real line. Replacing θ by z, we obtain $t(z) := \sum_{\nu=-n}^{n} a_\nu \mathrm{e}^{\mathrm{i}\nu z}$, which is holomorphic throughout the complex plane. Thus, a trigonometric polynomial $t(\theta)$ can be seen as the restriction of an entire function to the real axis. Unless all the coefficients a_ν except a_0 are zero, $t(z)$ is an entire function of order 1 and of type $T \le n$. Clearly, there exists a constant C such that $|t(z)| < C\,\mathrm{e}^{n|z|}$ for all $z \in \mathbb{C}$. In other words, $t(z)$ is an entire function of exponential type n. Let us recall that a function $f(z)$ holomorphic in an unbounded domain $\mathcal{D} \subseteq \mathbb{C}$ is said to be of exponential type τ in $\mathcal{D}$ if for any $\varepsilon > 0$, there exists a constant $K(\varepsilon)$ such that $|f(z)| < K(\varepsilon)\,\mathrm{e}^{(\tau+\varepsilon)|z|}$ for all $z \in \mathcal{D}$. In the present context, an interesting example of an unbounded domain is the sector

$$A(\alpha, \beta) := \{z = r\mathrm{e}^{\mathrm{i}\theta} : 0 < r < \infty\,,\ \alpha \le \theta \le \beta\}\,,$$

where $\beta \in (\alpha\,,\ \alpha + 2\pi)$, and half-planes have special significance. Some of the important results about functions of exponential type are to be found in what follows.

We know that trigonometric polynomials are 2π-periodic, but an entire function of exponential type may not be periodic at all; $(\sin \tau z)/z$ is such a function. As another example, we wish to mention

$$f(z) := \sum_{\nu=0}^{n} a_\nu e^{i\lambda_\nu z}\,, \ \lambda_0 < \cdots < \lambda_n\,, \tag{2.1}$$

which is an entire function of exponential type $\tau := \max\{|\lambda_0|\,, |\lambda_n|\}$ but is generally not periodic.

It is known (see [2, Theorem 6.10.1]) that if $f(z)$ is an entire function of exponential type τ which is periodic on the real axis with period Δ, then it must be of the form $f(z) = \sum_{\nu=-n}^{n} a_\nu\, e^{2\pi i\nu z/\Delta}$ with $n \le \lfloor \Delta\tau/(2\pi)\rfloor$.

To characterize the dependence of the growth of a function f of exponential type τ in a sector $A(\alpha,\beta)$ on the direction in which z tends to infinity, Phragmén and Lindelöf introduced the function

$$h_f(\theta) := \limsup_{r\to\infty} \frac{\log|f(r\,e^{i\theta})|}{r} \qquad (\alpha \le \theta \le \beta)\,,$$

called the *indicator function* of f. It is known that unless $h_f(\theta) \equiv -\infty$, $h_f(\theta)$ is continuous in $\alpha < \theta < \beta$ and that if $\alpha \le \theta < \theta + \pi \le \beta$, then

$$h_f(\theta) + h_f(\theta + \pi) \ge 0\,. \tag{2.2}$$

If f is an entire function of order 1 whose type is τ, then, $h_f(\theta) \le \tau$ for all θ and so, by (2.2), $h_f(\theta) \ge -\tau$. See [2, Chapter 5] for these and many other properties of the indicator function.

Bernstein's inequality for entire functions of exponential type. Bernstein himself was the first to extend inequality (1.1) to entire functions of exponential type. The extended version may be stated as follows.

Theorem A (S. Bernstein [9, p. 513]). *Let $f(z)$ be an entire function of exponential type τ such that $|f(x)| \le M$ on the real axis. Then*

$$|f'(x)| \le M\tau \qquad (x \in \mathbb{R})\,. \tag{2.3}$$

In (2.3) equality holds if and only if $f(z)$ is of the form $a\,e^{-i\tau z} + b\,e^{i\tau z}$, where $|a| + |b| = M$.

If $P(z)$ is a polynomial of degree at most n, then $f(z) := P(e^{iz})$ is an entire function of exponential type n. Besides, $|f(x)| \le M$ on the real axis if $|P(z)| \le M$ on the unit circle. Hence, inequality (1.1) is covered by (2.3).

A basic lemma. The following lemma [2, Theorem 6.2.4] serves as a basic tool in the study of functions of exponential type. In [8] the reader will find a proof of this result, which contains a thorough discussion of the case of equality.

Lemma A. *Let f be a function of exponential type in the open upper half-plane such that $h_f(\pi/2) \le c$. Furthermore, let f be continuous in the closed upper half-plane and suppose that $|f(x)| \le M$ on the real axis. Then*

$$|f(x+\mathrm{i}y)| < M\,\mathrm{e}^{cy} \qquad (-\infty < x < \infty,\ y > 0) \tag{2.4}$$

unless $f(z) \equiv M\,\mathrm{e}^{\mathrm{i}\gamma}\,\mathrm{e}^{-\mathrm{i}cz}$ for some real γ.

3. Entire functions of exponential type satisfying $f(z) \neq 0$ in $\Im z > k$, $k > 0$. Now, we shall formulate and prove an extension of a theorem of N. K. Govil to entire functions of exponential type. Govil's theorem [6, p. 52] may be stated as follows.

Theorem B. *Let $P(z)$ be a polynomial of degree n having no zeros in the open disk $|z| < k$, $k \le 1$, and let $Q(z) := z^n\overline{P(1/\overline{z})}$. If $\max_{|z|=1}|P'(z)|$ and $\max_{|z|=1}|Q'(z)|$ are attained at the same point of the unit circle $|z| = 1$, then*

$$\max_{|z|=1}|P'(z)| \le \frac{n}{1+k^n}\max_{|z|=1}|P(z)|\,. \tag{3.1}$$

The bound is attained for the polynomial $P(z) := c\,(z^n + k^n)$, $c \in \mathbb{C}$.

Our result may be stated as follows.

Theorem 1. *Let $f(z)$ be an entire function of order 1 and type τ having no zeros in the half-plane $\Im z > k$ for some $k > 0$. In addition, let $h_f(\pi/2) = 0$ and $|f(x)| \le M$ on the real axis. Define $\omega_f(z) := \mathrm{e}^{\mathrm{i}\tau z}\,\overline{f(\overline{z})}$ and suppose that*

$$\sup_{-\infty<x<\infty}|f'(x)| \quad \textit{and} \quad \sup_{-\infty<x<\infty}|\omega_f'(x)|$$

are both attained at the same point of the real axis. Then

$$|f'(x)| \le \frac{M\tau}{1+\mathrm{e}^{-\tau k}} \qquad (-\infty < x < \infty)\,. \tag{3.2}$$

The following special case of Theorem 1 deserves to be mentioned explicitly. For basic facts about uniformly almost periodic functions, we refer the reader to [1] or [4].

Corollary 1. *Let $f(z)$ be a uniformly almost periodic entire function of exponential type τ having no zeros in the half-plane $\Im z > k$ for some $k > 0$ and let $h_f(\pi/2) = 0$. In addition, let $|f(x)| \le M$ on the real axis and suppose that the Fourier coefficients of f are all non-negative. Then* (3.2) *holds.*

4. Some more auxiliary results. The proof of Theorem 1 requires some preparation which consists in recalling certain notions and additional results about entire functions of exponential type.

Definition 1. An entire function f of exponential type is said to belong to the *class* $\mathbf{P}$ if it has no zeros in the open lower half-plane and $h_f(-\pi/2) \geq h_f(\pi/2)$.

Note. From (2.2) it follows that if $f \not\equiv 0$ is an entire function of exponential type 0, then $h_f(\theta) = 0$ for all θ. Hence, any entire function of exponential type 0 having all its zeros in the closed upper half-plane belongs to the class $\mathbf{P}$.

It is known (see [7] or [2, Theorem 7.8.3]) that the Hadamard factorization of a function f belonging to the class $\mathbf{P}$ has the form

$$f(z) = A\, z^m\, \mathrm{e}^{cz} \prod_{k=1}^{\infty} \left\{ \left(1 - \frac{z}{z_k}\right) \mathrm{e}^{z\, \Re(1/z_k)} \right\}, \tag{4.1}$$

where $z_k \neq 0$, $\Im z_k \geq 0$ and $2\Im c = h_f(-\pi/2) - h_f(\pi/2) \geq 0$.

It is also known [2, p. 129, Theorem 7.8.1] that if f belongs to $\mathbf{P}$, then

$$|f(z)| \geq |f(\overline{z})| \qquad (\Im z < 0)\,. \tag{4.2}$$

From (4.2) it follows that if f belongs to $\mathbf{P}$, then $h_f(-\alpha) \geq h_f(\alpha)$ for all $\alpha \in (0\,,\,\pi)$.

The following result (see [7, p. 59, Lemma 3] or [2, p. 130, Theorem 7.8.6]) is of fundamental importance. Its significance in the present context cannot be over-emphasized.

Lemma B. *Let f be an entire function of order* 1 *and type τ belonging to the class $\mathbf{P}$. Furthermore, let g be an entire function of exponential type $\sigma \leq \tau$ such that*

$$|g(x)| \leq |f(x)| \quad \textit{for all} \quad x \in \mathbb{R}\,. \tag{4.3}$$

Then $\phi_\lambda(z) := g(z) - \lambda f(z)$ belongs to $\mathbf{P}$ for any $\lambda \in \mathbb{C}\,,\ |\lambda| > 1$.

Definition 2. An additive homogeneous operator $B[f(z)]$ which carries entire functions of exponential type into entire functions of exponential type and leaves the class $\mathbf{P}$ invariant is called (see [7, p. 60] or [2, p. 225, Definition 11.7.1]) a B-operator.

It may be added that an operator B is *additive* if $B[f+g] = B[f] + B[g]$ and *homogeneous* if $B[cf] = cB[f]$.

Using the representation (4.1), it can be easily shown that *differentiation* is also a B-operator (see [2, p. 226]).

Let $f(z)$ be an entire function of order 1 and type τ. Suppose that $|f(x)| \leq M$ on the real axis and that $h_f(\pi/2) \leq 0$. Then by Lemma A, $|f(z)| < M$ in the open upper half-plane. Hence, $\phi(z) := f(z) - M\mathrm{e}^{-\mathrm{i}\alpha}$, $\alpha \in \mathbb{R}$, is an entire function of order 1 and type τ which has no zeros in the open upper half-plane. Consequently, the function

$$\omega_\phi(z) := \mathrm{e}^{\mathrm{i}\tau z}\, \overline{\phi(\overline{z})} = \omega_f(z) - M\mathrm{e}^{\mathrm{i}\alpha}\, \mathrm{e}^{\mathrm{i}\tau z}$$

belongs to the class **P** and $|\phi(x)| = |\omega_\phi(x)|$ for all real x. By Lemma B, the function $\phi(z) - \lambda\,\omega_\phi(z)$ belongs to the class **P** for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Since differentiation is a B-operator, the function $\phi'(z) - \lambda\,\omega'_\phi(z)$ also belongs to the class **P** for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. In particular, $\phi'(z) - \lambda\,\omega'_\phi(z) \neq 0$ in the lower half-plane for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. In other words,

$$f'(z) - \lambda\left(\omega'_f(z) - M\,\mathrm{i}\tau\,\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\tau z}\right) \neq 0 \tag{4.4}$$

for any z with $\Im z < 0$, for any $\alpha \in \mathbb{R}$ and for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Now, note that f is not a constant and so $\omega_f(z)$ cannot be of the form $M\,\mathrm{e}^{\mathrm{i}\gamma}\,\mathrm{e}^{\mathrm{i}\tau z}$, $\gamma \in \mathbb{R}$. Hence, by Theorem A and Lemma A, $\omega'_f(z) - M\,\mathrm{i}\tau\,\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\tau z}$ is different from zero at every point of the open lower half-plane. Hence (4.4) can hold for any z with $\Im z < 0$, any $\alpha \in \mathbb{R}$ and any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ only if

$$|f'(z)| \leq M\tau\mathrm{e}^{-\tau\Im z} - |\omega'_f(z)|\,.$$

Hence, the following result holds. Thus, we have proved that *if f is an entire function of order* 1 *and type τ such that $|f(x)| \leq M$ on the real axis and $h_f(\pi/2) \leq 0$, then*

$$|f'(z)| + |\omega'_f(z)| \leq M\,\tau\,\mathrm{e}^{-\tau\Im z} \qquad (\Im z < 0)\,.$$

By continuity, the same must be true for z belonging to the real axis. In other words, the following result holds.

Lemma B. *Let f be an entire function of order* 1 *and type τ. Suppose, in addition, that $|f(x)| \leq M$ on the real axis and that $h_f(\pi/2) \leq 0$. Then*

$$|f'(z)| + |\omega'_f(z)| \leq M\,\tau\,\mathrm{e}^{-\tau\Im z} \qquad (\Im z \leq 0)\,. \tag{4.5}$$

5. Proof of Theorem 1. As the first step towards the proof of Theorem 1, we prove the following proposition.

Proposition 1. *Let F be an entire function of order* 1 *and type τ having all its zeros in the half-plane $\{z \in \mathbb{C} : \Im z \geq -k\}$ for some $k > 0$. Suppose that $|F(x)|$ is bounded on the real axis and that $h_F(\pi/2) \leq 0$. In addition, let $\omega_F(z) := \mathrm{e}^{\mathrm{i}\tau z}\overline{F(\overline{z})}$. Then*

$$\sup_{-\infty<x<\infty} |\omega'_F(x)| \leq \mathrm{e}^{\tau k} \sup_{-\infty<x<\infty} |F'(x)|\,. \tag{5.1}$$

Proof. Suppose that $|F(x)| \leq M$ on the real axis. The function defined by $g(z) := F(z - \mathrm{i}k)$ is of order 1 and type τ. Besides, by Lemma A, $|g(x)| \leq M\,\mathrm{e}^{\tau k}$ for all real x. We claim that g belongs to the class **P** introduced in Definition 1. Clearly, g has no zeros in the open lower half-plane. Hence, it is sufficient to check that $h_g(-\pi/2) \geq h_g(\pi/2)$.

Since $|g(x)|$ is bounded on the real axis and $h_g(\pi/2) = h_F(\pi/2) \le 0$, we must necessarily have

$$h_g\left(-\frac{\pi}{2}\right) = h_F\left(-\frac{\pi}{2}\right) = \tau\,,$$

otherwise, by Lemma A, g and so F would not be of order 1 and type τ. Note that τ must be positive because a function of order 1 that is bounded on the real axis or on any line cannot be of type 0. Thus, $h_g(-\pi/2) > 0$ whereas $h_g(\pi/2) \le 0$. Hence in fact, $h_g(-\pi/2) > h_g(\pi/2)$ and so g belongs to **P**.

Let $\omega_g(z) := \mathrm{e}^{\mathrm{i}\tau z}\overline{g(\overline{z})}$. Then, $|\omega_g(x)| = |g(x)| \le M\,\mathrm{e}^{\tau k}$ for all real x. Besides, $h_{\omega_g}(\pi/2) = -\tau + h_g(-\pi/2) = 0$. Hence, by Lemma A, $|\omega_g(z)| \le M\,\mathrm{e}^{\tau k}$ in the upper half-plane. Since

$$\begin{aligned}\omega_g(z) &= \mathrm{e}^{\mathrm{i}\tau z}\overline{F(\overline{z} - \mathrm{i}k)} \\ &= \mathrm{e}^{\tau k}\,\mathrm{e}^{\mathrm{i}\tau(z+\mathrm{i}k)}\,\overline{F(\overline{z + \mathrm{i}k})} = \mathrm{e}^{\tau k}\omega_F(z+\mathrm{i}k)\end{aligned}$$

we see that

$$h_{\omega_g}\left(-\frac{\pi}{2}\right) = \tau + h_g\left(\frac{\pi}{2}\right) \le \tau$$

and so, by Lemma A, $|\omega_g(z)| \le M\,\mathrm{e}^{\tau(k+|\Im z|)}$ in the lower half-plane. In particular, $\omega_g(z)$ is an entire function of exponential type at most τ.

We have a function g of order 1 and type τ which belongs to the class **P**. Besides, we have a function $\omega_g(z)$ of exponential type τ such that $|\omega_g(x)| = |g(x)|$ for all real x. So, Lemma B may be applied with g in place of f and ω_g in place of g to conclude that for any λ such that $|\lambda| > 1$, the function $\omega_g(z) - \lambda\,g(z)$ belongs to the class **P**. Since differentiation is a B-operator, the function $\omega_g'(z) - \lambda\,g'(z)$ also belongs to the class **P** for any $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$. In particular, $\omega_g'(z) - \lambda\,g'(z) \ne 0$ if $\Im z < 0$ for any $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$. This is possible only if $|\omega_g'(z)| \le |g'(z)|$ for any z in the open lower half-plane. By continuity, the same must be true for any real z also. Thus, $|\omega_g'(z)| \le |g'(z)|$ for $\Im z \le 0$, which means that

$$\mathrm{e}^{\tau k}\,|\omega_F'(z+\mathrm{i}k)| \le |F'(z-\mathrm{i}k)| \qquad (\Im z \le 0)\,.$$

Taking $z = x - \mathrm{i}k$, in this inequality, we obtain

$$\mathrm{e}^{\tau k}\,|\omega_F'(x)| \le |F'(x - 2\mathrm{i}k)| \qquad (-\infty < x < \infty)\,. \tag{5.2}$$

Since F is an entire function of order 1 and type τ, the same can be said about the function F'. Hence, by Lemma A, applied to the function $\overline{F'(\bar{z})}$, we obtain

$$|F'(x - 2\mathrm{i}k)| \le \mathrm{e}^{2\tau k} \sup_{-\infty<x<\infty} |F'(x)|$$

for any real x. Combining this with (5.2), we find that

$$|\omega_F'(x)| \le \mathrm{e}^{\tau k} \sup_{-\infty<x<\infty} |F'(x)|$$

for any real x, which is equivalent to (5.1). $\square$

Proposition 2. *Let f be an entire function of order 1 and type τ having no zeros in the half-plane $\Im z > k$ where $k > 0$. Besides, let $h_f(\pi/2) = 0$ and suppose that $|f(x)|$ is bounded on the real axis. In addition, let $\omega_f(z) := \mathrm{e}^{\mathrm{i}\tau z}\overline{f(\overline{z})}$. Then*

$$\mathrm{e}^{-\tau k} \sup_{-\infty<x<\infty} |f'(x)| \leq \sup_{-\infty<x<\infty} |\omega'_f(x)| . \tag{5.3}$$

Proof. Lemma A can be used to see that $h_f(-\pi/2) = \tau$. Hence, $\omega_f(z) := \mathrm{e}^{\mathrm{i}\tau z}\overline{f(\overline{z})}$ is an entire function of order 1 and type τ having all its zeros in the half-plane $\Im z > -k$. Besides, $h_{\omega_f}(\pi/2) = 0$ and $|\omega_f(x)|$ is bounded on the real axis. Hence, ω_f satisfies all the conditions of Proposition 1. So, let us apply Proposition 1 taking $F = \omega_f$. Clearly, then $\omega_F = \omega_{\omega_f} = f$ and so by (5.1), we have

$$\sup_{-\infty<x<\infty} |f'(x)| \leq \mathrm{e}^{\tau k} \sup_{-\infty<x<\infty} |\omega'_f(x)| ,$$

which proves (5.3). $\square$

Proof of Theorem 1. Suppose that

$$\sup_{-\infty<x<\infty} |f'(x)| \quad \text{and} \quad \sup_{-\infty<x<\infty} |\omega'_f(x)|$$

are both attained at the same point x_0 of the real axis. Combining (4.5) and (5.3), we obtain that

$$\begin{aligned}(1 + \mathrm{e}^{-\tau k}) \sup_{-\infty<x<\infty} |f'(x)| &\leq \sup_{-\infty<x<\infty} |f'(x)| + \sup_{-\infty<x<\infty} |\omega'_f(x)| \\ &\leq |f'(x_0)| + |\omega'_f(x_0)| \\ &\leq M\tau.\end{aligned}$$

Then

$$\sup_{-\infty<x<\infty} |f'(x)| \leq \frac{\tau}{1 + \mathrm{e}^{-\tau k}} M,$$

which proves the theorem. $\square$

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