



## Two hierarchies of $\mathbf{R}$ -recursive functions

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### Abstract

In the paper some aspects of complexity of  $\mathbf{R}$ -recursive functions are considered. The limit hierarchy of  $\mathbf{R}$ -recursive functions is introduced by the analogy to the  $\mu$ -hierarchy. Then its properties and relations to the  $\mu$ -hierarchy are analysed.

### 1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorzczuk hierarchy, compare [1]).

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorzczuk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals  $\mathbf{R}$  (called  $\mathbf{R}$ -recursive functions) in the analogous way to the classical recursive functions on the natural numbers  $\mathbf{N}$ . His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation  $\mu$ , which is used to construct  $\mu$ -hierarchy of  $\mathbf{R}$ -recursive functions.

It was shown [5] that the zero-finding operator  $\mu$  can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to  $\mu$ -hierarchy.

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## 2. Preliminaries

We start with a fundamental definition of a class of real functions called **R**-recursive functions [4].

**Definition 2.1** *The set of **R**-recursive functions is generated from the constants 0,1 by the operations:*

- 1) *composition:  $h(\bar{x}) = f(g(\bar{x}))$ ;*
- 2) *differential recursion:  $h(\bar{x}, 0) = f(\bar{x}), \partial_y h(\bar{x}, y) = g(\bar{x}, y, h(\bar{x}, y))$  (the equivalent formulation can be given by integrals:  $h(\bar{x}, y) = f(\bar{x}) + \int_0^y g(\bar{x}, y', h(\bar{x}, y')) dy'$ );*
- 3) *m-recursion  $h(\bar{x}) = m_y f(\bar{x}, y) = \inf \{y : f(\bar{x}, y) = 0\}$ , where infimum chooses the number **y** with the smallest absolute value and for two **y** with the same absolute value the negative one;*
- 4) *vector-valued functions can be defined by defining their components.*

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if **h** is defined by a differential recursion then **h** is defined only where a finite and unique solution exists. This is why the set of **R**-recursive functions includes also partial functions. We use (after [4]) the name of **R**-recursive functions in the article, however we should remember that in reality we have partiality here (partial **R**-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive **y** or just below some negative **y** then the infimum operation returns that **y** even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] **m**-operation is replaced by infinite limits:  $h(\bar{x}) = \liminf_{y \rightarrow \infty} g(\bar{x}, y)$ ,  $h(\bar{x}) = \limsup_{y \rightarrow \infty} g(\bar{x}, y)$  then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form  $h(\bar{x}) = \lim_{y \rightarrow \infty} g(\bar{x}, y)$ , which can be in the obvious way obtained from limsup, liminf:

**Corollary 2.2** *The class of  $\mathbf{R}$ -recursive functions is closed under the operations of infinite limits:  $h(\bar{x}) = \liminf_{y \rightarrow \infty} g(\bar{x}, y)$ ,  $h(\bar{x}) = \limsup_{y \rightarrow \infty} g(\bar{x}, y)$ ,  $h(\bar{x}) = \lim_{y \rightarrow \infty} g(\bar{x}, y)$ .*

### 3. Hierarchies

The operator  $m$  is a key operator in generating the  $\mathbf{R}$ -recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of  $m$  in the definition of a given  $f$ .

**Definition 3.1** ([4]) *For a given  $\mathbf{R}$ -recursive expression  $s(\bar{x})$ , let  $M_{x_i}(s)$  (the  $m$ -number with respect to  $x_i$ ) be defined as follows:*

$$M_x(0) = M_x(1) = M_x(-1) = 0, \quad (1)$$

$$M_x(f(g_1, g_2, \dots)) = \max_j (M_{x_j}(f) + M_x(g_j)), \quad (2)$$

$$M_x\left(h = f + \int_0^y g(\bar{x}, y', h) dy'\right) = \max(M_x(f), M_x(g), M_h(g)), \quad (3)$$

$$M_y\left(h = f + \int_0^y g(\bar{x}, y', h) dy'\right) = \max(M_{y'}(g), M_h(g)), \quad (4)$$

$$M_x(m_y f(\bar{x}, y)) = \max(M_x(f), M_y(f)) + 1, \quad (5)$$

where  $x$  can be any  $x_1, \dots, x_n$  for  $\bar{x} = (x_1, \dots, x_n)$ .

For an  $\mathbf{R}$ -recursive function  $f$ , let  $M(f) = \max_{x_i}(s)$  minimized over all expressions  $s$  that define  $f$ . Now we are ready to define  $M$ -hierarchy ( $m$ -hierarchy) as a family of  $M_j = \{f : M'(f) \leq j\}$ .

Let us construct the analogous definition of  $L$ -hierarchy by replacing in the above definition  $M_x$  by  $L_x$  and changing line (5) to the following form (5'):

$$\begin{aligned} L_x\left(\liminf_{y \rightarrow \infty} g(\bar{x}, y)\right) &= L_x\left(\limsup_{y \rightarrow \infty} g(\bar{x}, y)\right) = \\ &= L_x\left(\lim_{y \rightarrow \infty} g(\bar{x}, y)\right) = \max(L_x(f), L_y(f)) + 1. \end{aligned}$$

For an  $\mathbf{R}$ -recursive function  $f$ , let  $L(f) = \max_i L_{x_i}(s)$  minimized over all expressions  $s$  that define  $f$  without using the  $m$ -operation.

**Definition 3.2** The **L**-hierarchy is a family of  $L_j = \{f : L(f) \leq j\}$ .

Let us add that in Definition 3.2 we use explicitly the operator  $f(\bar{x}) = \lim_{y \rightarrow \infty} g(\bar{x}, y)$  to avoid its construction by other operators (lim sup, lim inf), which would effect in a superficially higher class of a complexity of a function  $f$ .

As an obvious corollary from definitions we have the following statement.

**Lemma 3.3** The classes  $M_0$  and  $M_1$  are identical.

A function  $f \in L_0 = M_0$  will be called (by an analogy to the case of natural recursive functions) a primitive **R**-recursive function. After Moore [4] we can conclude that such functions as:  $-x$ ,  $x + y$ ,  $xy$ ,  $x/y$ ,  $e^x$ ,  $\ln x$ ,  $y^x$ ,  $\sin x$ ,  $\cos x$  are primitive **R**-recursive.

We can give a few results on some levels of the limit hierarchy.

**Lemma 3.4.** The Kronecker **d** function, the signum function and absolute value belong to the first level ( $L_1$ ) of limit hierarchy.

**Proof.** It is sufficient to take the following definitions [5]: hence  $d(0) = 1$  and

for all  $x \neq 0$  we have  $d(x) = 0$  let us define  $d(x) = \liminf_{y \rightarrow \infty} \left( \frac{1}{1+x^2} \right)^y$ . Now

from the expression  $\liminf_{y \rightarrow \infty} \arctan xy = \begin{cases} p/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -p/2, & \text{if } x < 0, \end{cases}$  we obtain

$$\operatorname{sgn}(x) = \frac{\liminf_{y \rightarrow \infty} \arctan xy}{2 \arctan 1} \text{ and } |x| = \operatorname{sgn}(x)x.$$

We should be careful with definitions of functions by cases:

**Lemma 3.5** For  $h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ \mathbf{M} & \mathbf{M} \\ g_k(\bar{x}), & \text{if } f(\bar{x}) \geq k-1 \end{cases}$  and  $g_i \in L_{n_i}$  for all  $1 \leq i \leq k$ ,

$f \in L_m$  the function  $h$  belongs to  $L_{\max(n_1, \dots, n_k, m+1)}$

**Proof.** Let us see that  $eq(x, y) = d(x - y) \in L_1$  and

$$ge(x, y) = \frac{(\text{sgn}(x - y) + eq(x, y))}{2} + \frac{1}{2} \in L_1. \quad \text{Then of course}$$

$$h(\bar{x}) = \sum_{i=1}^{k-1} g_i(\bar{x}) eq(f(\bar{x}), i-1) + g_k(\bar{x}) ge(f(\bar{x}), k-1). \quad \square$$

Of course this result can be easily extended to other forms of definitions by cases.

**Lemma 3.6** *The function  $\Theta(x)$  (equal to 1 if  $x \geq 0$ , otherwise 0), maximum  $\max(x, y)$ , square-wave function  $s$  are in  $L_2$ , the function  $p(x)$  such that  $p(x) = 1$  for  $x \in [2n, 2n+1]$  and  $p(x) = 0$  for  $x \in [2n+1, 2n+2]$  is in  $L_2$  and the floor function  $\lfloor x \rfloor$  is in  $L_3$ .*

**Proof.** We give the proper definitions (from [6]) for these functions. Let

$$\Theta(x) = d(x - |x|),$$

$$\max(x, y) = xd(x - y) + (1 - d(x - y))[x\Theta(x - y) + y\Theta(y - x)],$$

$$s(x) = \Theta(\sin(px)).$$

The function  $p(x)$  can be given as  $s(x) \left( 1 - d \left( \sin \left( \frac{(x-1)p}{2} \right) \right) \right)$ , so  $p \in L_2$ .

The floor function we can define by the auxiliary function  $w(0) = 0$ ,  $\partial_x w(x) = 2\Theta(-\sin(2px))$  as

$$\lfloor x \rfloor = \begin{cases} 2w(x/2) & \text{if } p(x) = 1, \\ 2w((x-1)/2) & \text{if } p(x) = 0. \end{cases}$$

From the above equation we have  $\lfloor x \rfloor$  in  $L_3$ .  $\square$

Let us recall that if  $f : R^n \rightarrow R$  is an  $\mathbf{R}$ -recursive function then the function  $f_{iter}(i, \bar{x})$  is  $\mathbf{R}$ -recursive, too.

**Lemma 3.7** *Let  $f : R^n \rightarrow R$  belongs to the class  $L_i$ , then we have  $f_{iter} : R^{n+1} \rightarrow R$  is in  $L_{\max(2, i)}$ .*

**Proof.** The definitions, which were given by Moore [3]  $f_{iter}(i, \bar{x}) = h(2i)$ , where

$$h(0) = g(0) = \bar{x},$$

$$\begin{aligned} \partial_t g(t) &= [f(h(t)) - h(t)]s(t), \\ \partial_t h(t) &\geq \left[ \frac{g(t) - h(t)}{r(t)} \right] (1 - s(t)), \end{aligned}$$

with  $s$  - a square wave function in  $L_2$  and  $r(0) = 0$ ,  $\partial_t r(t) = 2s(t) - 1$ ,  $r, s \in L_2$  give us the desirable statement.  $\square$

**Lemma 3.8** *The  $\mathbf{R}^1$ -recursive functions  $g_2 : \mathbf{R}^2 \rightarrow \mathbf{R}$ ,  $g_2^1, g_2^2 : \mathbf{R} \rightarrow \mathbf{R}$  such that  $(\forall x, y \in \mathbf{R}) g_2^1(g_2(x, y)) = x$ ,  $(\forall x, y \in \mathbf{R}) g_2^2(g_2(x, y)) = y$ , have the following properties:  $g_2, g_2^1$  are in  $L_{10}$ ,  $g_2^2$  is in  $L_{14}$ .*

**Proof.** We have the auxiliary functions  $\Gamma_2, \Gamma_2^1, \Gamma_2^2$ , which are coding and decoding functions in the interval  $(0, 1) : \Gamma_2(x, y) = c(x) + c(y)/10$ , where

$$c(x) = \lim_{i \rightarrow \infty} z(a(i, x))/10^{2i} + b(i, x)/10^i,$$

and later  $z(x) = \lim_{i \rightarrow \infty} z'_{iter}(i, x)$ ,

$$\begin{aligned} z'_{iter}(i, a_1 \dots a_n . a_{n+1} \dots) &= a_1 \dots a_n 0 \dots a_{n+1} 0 . a_{n+i+1} \dots, \\ a(i, 0 . a_1 a_2 \dots a_i \dots) &= 0 . a_1 \dots a_i \\ b(i, 0 . a_1 a_2 \dots a_i \dots) &= 0 . 0 \dots 0 a_{i+1} \dots, \end{aligned}$$

$(z'(x) = \begin{cases} 100 \lfloor x \rfloor + 10(x - \lfloor x \rfloor), & \text{if } \lfloor x \rfloor \neq x, \\ x, & \text{if } \lfloor x \rfloor = x; \end{cases} \in L_4, a, b \in L_4$ . Also  $z'_{iter}$  belongs

to  $L_4$ , hence  $\Gamma_2(x, y) \in L_{10}$ , decoding of the first element is described in the symmetric way so  $\Gamma_2^1(x)$  is in  $L_{10}$ , but  $\Gamma_2^2(x) = \Gamma_2^1(10 - \lfloor 10x \rfloor)$  so  $\Gamma_2^2 \in L_{14}$ .

The functions  $\Gamma_2, \Gamma_2^1, \Gamma_2^2$  can be extended to all reals by one-to-one  $f : (0, 1) \rightarrow \mathbf{R} \in L_0$  without the loss of their class.  $\square$

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions  $g_n : \mathbf{R}^n \rightarrow \mathbf{R}$  and  $g_n^i : \mathbf{R} \rightarrow \mathbf{R}$  for  $i = 1, \dots, n$  such that

$$(\forall i)(\forall x_1, \dots, x_n \in \mathbf{R}) g_n^i(g_n(x_1, \dots, x_n)) = x_i$$

in the same class:  $g_n, g_n^1 \in L_{10}$  and  $(\forall i > 1) g_n^i \in L_{14}$ .

We finish this part with the important form of defining: a new function is given as a product of values  $\mathbf{f}$  in some integer points.

**Lemma 3.9** *There exists such constant  $p \in \mathbb{N}$  that for the function*

$$\prod_{z=0}^y f(\bar{x}, z) = \begin{cases} f(\bar{x}, 0) f(\bar{x}, 1) \dots f(\bar{x}, \lfloor y-1 \rfloor), & \text{if } y \geq 1, \\ 1, & \text{if } 0 \leq y < 1, \\ 0, & \text{if } y < 0, \end{cases}$$

*if the function  $f$  is in the class  $L_m$  then  $\prod_{z=0}^y f(\bar{x}, z)$  is in the class  $L_{m+p}$  ( $p$  is independent of  $m$ ).*

**Proof.** By the definitions

$$t(w) = g_{n+2}(g_{n+2}^{1,n}(w), g_{n+2}^{n+1}(w) + 1, f(g_{n+2}^{1,n}(w), g_{n+2}^{n+1}(w)) \cdot g_{n+2}^{n+2}(w))$$

and

$$S(\bar{x}, z) = t_{\lfloor z \rfloor} \dots t_{\lfloor z \rfloor} (s(\bar{x}, 0)) \dots = t_{iter}(\lfloor z \rfloor, g_{n+2}(\bar{x}, 0, 1))$$

we get the property

$$\prod_{y=0}^z f(\bar{x}, y) = g_{n+2}^{n+2}(S(\bar{x}, z)).$$

From the definition of the limit hierarchy we get  $\prod_{y=0}^z f(\bar{x}, y) \in L_{m+38}$ .  $\square$

In the rest of the paper we will use the constant  $p$  as the number of limits used in the recursive definition of the product  $\prod_{y=0}^z f(\bar{x}, y)$  instead of the value 38.

The above constructions are tedious and can be improved with a better approximation of  $p$ .

#### 4. Main results

Now we are ready to formulate two theorems which demonstrate connections between L-hierarchy and M-hierarchy.

**Theorem 4.1** *Let  $f : R^n \rightarrow R$  be an  $\mathbf{R}$ -recursive function. Then if  $f \in L_i$  then  $f \in M_{10i}$ .*

**Proof.** We use a simple induction here. The case  $i=0$  is given in Lemma 3.3. Now let us suppose that the thesis is true for  $i=n$ . Let  $f \in L_{n+1}$  be defined as  $f(\bar{x}) = \lim_{y \rightarrow \infty} g(\bar{x}, y)$  for  $g \in L_n$ . Then we can recall Theorem 4.2 from [6] which gives us the following result: to define  $\mathbf{f}$  from  $g$  it is necessary to use at

most 10  $m$ -operation. Hence for  $g \in M_{10n}$  the function  $f$  satisfies  $f \in M_{10n+10}$ . Similar inferences hold for  $\liminf, \limsup$ .  $\square$

Now we can give the result about the 'limit complexity' of the infimum operator  $m$

**Lemma 4.2** *If  $f(\bar{x}, y): R^{n+1} \rightarrow R$  is in the class  $L_m$  then the function  $g: R^n \rightarrow R, g(\bar{x}) = m_y f(\bar{x}, y)$  is in the class  $L_{m+3p+9}$  is from Lemma 3.9.*

**Proof.** Here we must employ the results from [6]. There we defined the function  $g: R^n \rightarrow R, g(\bar{x}) = m_y f(\bar{x}, y)$  for  $f(\bar{x}, y): R^{n+1} \rightarrow R$  ( $f$  -  $\mathbf{R}$ -recursive) replacing the  $m$ -operator by limit operation. First we introduced the function

$$Z^f(\bar{x}, z) = \begin{cases} \inf_y \{f: K^f(\bar{x}, y) = 0\}, & \text{if } z = 0 \text{ and } \exists y K^f(\bar{x}, y) = 0, \\ \text{undefined} & \text{if } z = 0 \text{ and } \forall y K^f(\bar{x}, y) \neq 0, \\ 1 & \text{if } z \neq 0, \end{cases}$$

given in the following way:

$$Z^f(\bar{x}, z) = \begin{cases} \text{undefined} & \text{if } (z = 0) \wedge (S^f(\bar{x}) < 1/12), \\ \sqrt{S^f(\bar{x}) - 1/12}, & \text{if } (z = 0) \wedge (S^f(\bar{x}) \geq 1/12) \\ & \wedge f(\bar{x}, \sqrt{S^f(\bar{x}) - 1/12}) = 0, \\ -\sqrt{S^f(\bar{x}) - 1/12}, & \text{if } (z = 0) \wedge (S^f(\bar{x}) \geq 1/12) \\ & \wedge f(\bar{x}, -\sqrt{S^f(\bar{x}) - 1/12}) = 0, \\ 1, & \text{if } z \neq 0. \end{cases}$$

where  $S^f(\bar{x}) = \lim_{t \rightarrow \infty} S_1^f(\bar{x}, t) + \lim_{t \rightarrow \infty} S_2^f(\bar{x}, t)$ . Both functions  $S_1^f, S_2^f$  are defined by an integration

$$S_i^f(\bar{x}, t) = \int y^2 \left( 1 - h^f(\bar{x}, (-1)^{i+1} y - 1/2, (-1)^{i+1} y + 1/2) \right) dy, \quad i = 1, 2$$

from  $h^f(\bar{x}, a, b) = \liminf_{t \rightarrow \infty} \prod_{w=0}^{z+1} K^f\left(\bar{x}, a + w \frac{b-a}{z}\right)$  where  $K^f$  is the characteristic function of  $f$ .

Hence we can conclude that if  $K^f$  is in the  $L_s$  then  $Z_f$  is in the class  $L_{s+p+3}$ . Let us finish with the definition of the characteristic function of the infimum of zeros of  $f$  (see Theorem 4.2 from [5])



$$K_m^f(y) = 1 - \lim_{a \rightarrow -\infty} \lim_{b \rightarrow \infty} \lim_{z \rightarrow \infty} G^f(\bar{x}, z, a, b, y),$$

where  $G^f(\bar{x}, z, a, b, y)$  divides the interval  $[a, b]$  into  $2^{\lfloor z \rfloor}$  equal subintervals and gives the value 1 for  $y$  from the subintervals, which contains the least zero of  $f$  in  $[a, b]$  and value 0 otherwise. Precisely for  $y$  from  $\left[ a, a + \frac{b-a}{2^{\lfloor z \rfloor}} \right]$

$$G^f(\bar{x}, z, a, b, y) = \begin{cases} 1, & \text{if } h^f\left(\bar{x}, a, a + \frac{b-a}{2^{\lfloor z \rfloor}}\right) = 0, \\ 0, & \text{otherwise} \end{cases}$$

for  $y \in \left( a + \frac{(k-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{k(b-a)}{2^{\lfloor z \rfloor}} \right)$  (where  $k = 2, 3, \dots, 2^n$ ) we have:

$$G^f(\bar{x}, z, a, b, y) = \begin{cases} 1, & \text{if } \prod_{i=1}^{k-1} h^f\left(\bar{x}, a + \frac{(i-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{i(b-a)}{2^{\lfloor z \rfloor}}\right) \neq 0 \\ \wedge h^f\left(\bar{x}, a + \frac{(k-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{k(b-a)}{2^{\lfloor z \rfloor}}\right) = 0, \\ 0, & \text{otherwise} \end{cases}$$

and for  $Y \notin [A, B]$  the function  $g_x^f$  is equal to 2.

The definition of  $G_f$  is given by the cases with respect to the value of the expression given by  $\prod h^f$ , since for  $f \in L_m$ , the function  $h_f \in L_{m+p+2}$  and  $G^f \in L_{m+2p+3}$ . Then we have  $K_m^f \in L_{m+2p+6}$ . Now we must use the function  $K_m^f$  in the same way as  $K^f$  which gives us  $Z_f$  in the class  $L_{m+3p+9}$ . The final definition of  $g(\bar{x}) = m_y f(\bar{x}, y)$  ([5] Theorem 4.3) given below

$$g(\bar{x}) = \begin{cases} Z^{f^+}(\bar{x}, 0) - Z^{f^-}(\bar{x}, 0), & \text{if } S^{f^+}(\bar{x}) < \frac{1}{12} \wedge S^{f^-}(\bar{x}) < \frac{1}{12}, \\ Z^{f^+}(\bar{x}, 0), & \text{if } \left( S^{f^+}(\bar{x}) \geq \frac{1}{12} \wedge S^{f^-}(\bar{x}) < \frac{1}{12} \right) \\ & \text{or} \\ & \left( S^{f^+}(\bar{x}) < \frac{1}{12} \wedge S^{f^-}(\bar{x}) < \frac{1}{12} \right) \\ & \wedge Z^{f^+}(\bar{x}, 0) < Z^{f^-}(\bar{x}, 0), \\ -Z^{f^-}(\bar{x}, 0), & \text{if } \left( S^{f^+}(\bar{x}) < \frac{1}{12} \wedge S^{f^-}(\bar{x}) \geq \frac{1}{12} \right) \\ & \text{or} \\ & \left( S^{f^+}(\bar{x}) < \frac{1}{12} \wedge S^{f^-}(\bar{x}) < \frac{1}{12} \right) \\ & \wedge Z^{f^+}(\bar{x}, 0) \geq Z^{f^-}(\bar{x}, 0), \end{cases}$$

where  $f^+(\bar{x}, y) = \begin{cases} f(\bar{x}, y), & y \geq 0, \\ 1, & y < 0; \end{cases}$   $f^-(\bar{x}, y) = \begin{cases} f(\bar{x}, -y), & y > 0, \\ 1, & y \leq 0; \end{cases}$  remains the class of  $g$  identical to the class of  $Z^f$ , i.e.  $g \in L_{m+3p+9}$ .  $\square$

**Theorem 4.3** Let  $f : R^n \rightarrow R$  be an  $\mathbf{R}$ -recursive function. Then for all  $i \geq 0$  if  $f \in M_i$  then  $f \in L_{(3p+9)i}$ .

The above statement is a simple consequence of the fact  $M_0 = L_0$  and Lemma 4.2.

## 5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the  $m$ -operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a  $m$ -operator.

We also establish the proper relation between the levels of the limit hierarchy and  $m$ -hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also  $m$ -hierarchy) and Baire classes

[7]. Also the kind of a connection between the  $\sum_n^0$ -measurable functions and  $\mathbf{R}$ -recursive functions is an open problem.

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