



## Newton-like method for singular 2-regular system of nonlinear equations

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### Abstract

In this article the problem of solving a system of singular nonlinear equations will be discussed. The theory of local and Q-superlinear convergence for the nonlinear operators is developed.

### 1. Introduction

Let  $F : D \subset R^n \rightarrow R^m$  be a nonlinear operator. The problem of solving a system of nonlinear equations consist in finding a solution  $x^* \in D$  of the equation

$$F(x) = 0. \quad (1)$$

#### Definition 1

A linear operator  $\Psi_2(h) : R^n \rightarrow R^m$ ,  $h \in R^n$  is called 2-factoroperator, if

$$\Psi_2(h) = F'(x^*) + P^\perp F''(x^*)h, \quad (2)$$

where

$P^\perp$  - denotes the orthogonal projection on  $(\text{Im } F'(x^*))^\perp$  in  $R^n$  [1].

#### Definition 2

Operator  $F$  is called 2-regular in  $x^*$  on the element  $h \in R^n$ ,  $h \neq 0$ , if the operator  $\Psi_2(h)$  has the property:

$$\text{Im } \Psi_2(h) = R^m.$$

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**Definition 3**

Operator  $F$  is called 2-regular in  $x^*$ , if  $F$  is 2-regular on the set  $K_2(x^*) \setminus \{0\}$ , where

$$K_2(x^*) = \text{Ker}F'(x^*) \cap \text{Ker}^2 P^\perp F''(x^*), \quad (3)$$

$$\text{Ker}^2 P^\perp F''(x^*) = \{h \in R^n : P^\perp F''(x^*)[h]^2 = 0\}.$$

We need the following assumption on  $F$ :

A1) completely degenerated in  $x^*$ :

$$\text{Im} F'(x^*) = 0. \quad (4)$$

A2) operator  $F$  is 2-regular in  $x^*$ :

$$\text{Im} F''(x^*)h = R^m \quad \text{for } h \in K_2(x^*), h \neq 0. \quad (5)$$

$$\text{A3) } \text{Ker}F''(x^*) \neq \{0\}. \quad (6)$$

If  $F$  satisfies A1 in  $x^*$ , then

$$K_2(x^*) = \text{Ker}^2 F''(x^*) = \{h \in R^n : F''(x^*)[h]^2 = 0\}. \quad (7)$$

In [1] it was proved, that if  $n=m$ , then the sequence

$$x_{k+1} = x_k - \left\{ \hat{F}'(x_k) + P_k^\perp F''(x_k)h_k \right\}^{-1} \cdot \left\{ F(x_k) + P_k^\perp F'(x_k)h_k \right\}, \quad (8)$$

where

$P_k^\perp$  - denotes orthogonal projection on  $(\text{Im} \hat{F}'(x_k))^\perp$  in  $R^n$ ,

$$h_k \in \text{Ker} \hat{F}'(x_k), \quad \|h_k\| = 1$$

converges Q-quadratically to  $x^*$ .

The matrices  $\hat{F}'(x_k)$  obtained from  $F'(x_k)$  by replacing all elements, whose absolute values do not increase  $v>0$ , by zero, where  $n = n_k = \|F(x_k)\|^{(1-\alpha)/2}$ ,  $0<\alpha<1$ .

In the case  $n = m+1$  the operator

$$\left\{ \hat{F}'(x_k) + P_k^\perp F''(x_k)h_k \right\}^{-1}$$

in method (8) is replaced by the operator

$$\left[ \hat{F}'(x_k) + P_k^\perp F''(x_k)h_k \right]^+ \quad (9)$$

and then the method converges Q-linearly to the set of solutions [2].

Under the assumptions A1-A3, the system of equation (1) is undetermined ( $n>m$ ) and degenerated in  $x^*$ .

## 2. Extending of the system of equation

Now we construct the operator  $\Phi : R^n \rightarrow R^{n-1}$  with the properties (4), (5) and such that  $\Phi(x^*)=0$  [2].

Assume

A4) Let  $F(x) = [f_1(x), f_2(x), \dots, f_m(x)]^T$ ,  $n > m$  is two continuously differentiable in some neighbourhood  $U \subset R^n$  of the point  $x^*$ .

Denote:

$$H = \text{lin}\{h\} \quad \text{for } h \in \text{Ker}^2 F'(x^*), h \neq 0.$$

$P = P_{H^\perp}$  denotes the orthogonal projection  $R^n$  on  $H^\perp$

$$f_i^0(x) = P(f_i'(x))^T \quad \text{for } i=1,2,\dots,m.$$

For each system of indices  $i_1, i_2, \dots, i_{n-m-1} \subset \{1, 2, \dots, m\}$  and vectors  $h_1, h_2, \dots, h_{n-m-1} \subset R^n$  we define

$$\Phi(x) = \begin{bmatrix} F'(x)h \\ j(x) \end{bmatrix}, \quad (10)$$

where

$$\begin{aligned} j(x) : R^n &\rightarrow R^r, \quad r = n-m-1, \\ j(x) &= PF'(x)h^0, \quad h^0 = [h_1, h_2, \dots, h_r]^T, \\ j(x) &= \begin{bmatrix} f_{i_1}^0(x)h_1 \\ \vdots \\ f_{i_r}^0(x)h_r \end{bmatrix}. \end{aligned} \quad (11)$$

In [2] it was proved, that the sequence

$$x_{k+1} = x_k - [\Phi'(x_k)]^+ \cdot \Phi(x_k), \quad k=0,1,2,\dots \quad (12)$$

quadratically converges to the solution of (1).

## 3. New method

We propose the Newton-like method, where the sequence  $\{x_k\}$  is defined by:

$$x_{k+1} = x_k - \{B_k\}^+ \cdot \Phi(x_k). \quad (13)$$

The operator  $\Phi'$  will be approximated by matrices  $\{B_k\}$ .

Let

$$s_k = x_{k+1} - x_k. \quad (14)$$

We propose matrices  $B_k$  which satisfy the secant equation:

$$B_{k+1}s_k = \Phi(x_{k+1}) - \Phi(x_k) \quad \text{for } k=0,1,2,\dots \quad (15)$$

For example, to obtain the sequence  $\{B_k\}$  we can apply the Broyden method:

$$B_{k+1} = B_k - \frac{r_k s_k^T}{s_k^T s_k} \quad \text{for } k=0,1,2,\dots \quad (16)$$

where

$$r_k = \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k. \quad (17)$$

We will prove for this method:

*Q-linear convergence* to  $x^*$  i.e. there exists  $q \in (0,1)$  such, that

$$\|x_{k+1} - x^*\| \leq q \|x_k - x^*\| \quad \text{for } k = 0,1,2,\dots \quad (18)$$

and next Q-superlinear convergence to  $x^*$ , i.e.:

$$\lim_{k \rightarrow \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0. \quad (19)$$

We present the theorem which is an analogue of the Bounded Deterioration Theorem (Broyden, Dennis and More - [3]) for the Newton-like methods, when the operator  $F'(x^*)$  is nonsingular.

### Theorem 1 (The Bounded Deterioration Theorem)

Let  $F$  satisfies the assumptions A1-A4. If exist constants  $q_1 \geq 0$  and  $q_2 \geq 0$  such that matrices  $\{B_k\}$  satisfy the inequality:

$$\|B_{k+1} - \Phi'(x^*)\| \leq (1 + q_1 r_k) \|B_k - \Phi'(x^*)\| + q_2 r_k, \quad (20)$$

then there are constants  $e > 0$  i  $d > 0$  such, that if

$$\|x_0 - x^*\| \leq e \quad \text{and} \quad \|B_0 - \Phi'(x^*)\| \leq d,$$

then the sequence

$$x_{k+1} = x_k - B_k^+ \Phi(x_k)$$

converges Q-linearly to  $x^*$ .

When the system of equation is rectangular, the proof of the theorem is analogous to that for the nonsingular and quadratic system and we neglect it.

### Theorem 2 (Linear convergence)

Let  $F$  satisfies the assumptions A1-A4. Then the method

$$x_{k+1} = x_k - \{B_k\}^+ \cdot \Phi(x_k),$$

$$B_{k+1} = B_k - \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k}$$

locally and Q-linearly converges to  $x^*$ .

### Proof.

To prove the Theorem we should prove the inequality (20) from Theorem 1. Now we notice:

$$\begin{aligned}
\|B_{k+1} - \Phi'(x^*)\| &= \left\| B_k - \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k} - \Phi'(x^*) \right\| \leq \\
&\leq \|B_k - \Phi'(x^*)\| + \left\| \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k} \right\| \leq \|B_k - \Phi'(x^*)\| + \\
&+ \left\| \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - \Phi'(x^*) s_k + \Phi'(x^*) s_k - B_k s_k\} s_k^T}{s_k^T s_k} \right\| \leq \|B_k - \Phi'(x^*)\| + \\
&+ \left\| \frac{(\Phi(x_{k+1}) - \Phi'(x^*)(x_{k+1} - x^*)) s_k^T}{s_k^T s_k} \right\| + \left\| \frac{(\Phi(x_k) - \Phi'(x^*)(x_k - x^*)) s_k^T}{s_k^T s_k} \right\| + \\
&+ \left\| \frac{(\Phi'(x^*) - B_k) s_k s_k^T}{s_k^T s_k} \right\| \leq \|\Phi'(x^*) - B_k\| (1 + q_1 r_k) + c_1 \frac{\|x_{k+1} - x^*\|^2 \|s_k\|}{\|s_k^T s_k\|} + \\
&+ c_2 \frac{\|x_k - x^*\|^2 \|s_k\|}{\|s_k^T s_k\|} \leq \|\Phi'(x^*) - B_k\| (1 + q_1 r_k) + q_2 r_k,
\end{aligned}$$

where  $c_1 > 0$ ,  $c_2 > 0$ ,  $q_1 > 0$ ,  $q_2 > 0$ ,  $r_k = \max\{\|x_{k+1} - x^*\|, \|x_k - x^*\|\}$ .

□

### Theorem 3 (Q-superlinear convergence)

Let  $F$  satisfies the assumptions A1-A4 and the sequence

$$\begin{aligned}
x_{k+1} &= x_k - \{B_k\}^{-1} \cdot \Phi(x_k), \\
B_{k+1} &= B_k - \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k}
\end{aligned}$$

linearly converges to  $x^*$ . Then the sequence  $\{x_k\}$  Q-superlinearly converges to  $x^*$ .

Proof.

Matrices  $B_k$  satisfy secant equation (15), so

$$B_{k+1} = P_{L_k}^\perp B_k \quad (21)$$

where

$$L_k = \{X : X s_k = y_k, \text{ where } y_k = \Phi'(x_{k+1}) - \Phi'(x_k)\} \quad (22)$$

Denote

$$H_k = H(x_k, x_{k+1}) = \int_0^1 \Phi'(x_k + t(x_{k+1} - x_k)) dt.$$

We have  $H_k \in L_k$  [4].

From (21) and [3] it follows:

$$\|B_{k+1} - B_k\|^2 + \|B_{k+1} - H_k\|^2 = \|B_k - H_k\|^2, \text{ for } i = 0, 1, 2, \dots.$$

By lemma 2 [5] we get  $\sum_{k=1}^{\infty} \|B_{k+1} - B_k\|^2 < \infty$ , thus we obtain

$$\|B_{k+1} - B_k\| \rightarrow 0.$$

This denotes that the method (13)-(17) is Q-superlinearly convergent [6], which ends the proof.  $\square$

#### 4. Summary

The proposed method is Q-superlinearly convergent and easier to apply than the method (12), without calculation of  $F''(x_k)$ .

#### References

- [1] Izmailov A.F., Tretiakov A.A., *Factor-analysis of non-linear mappings*, Moscow, Nauka, Fizmatlit Publishing Company, (1994) 336.
- [2] Chmura A., Tretiakov A.A., *New Methods for the solution of nonlinear equations with singularities*-Nonlinear Oscillations, 2 (1999) 1.
- [3] Broyden C.G., J.E. Dennis, J.J. More, *On the local and superlinear convergence of quasi-Newton methods*, Journal Inst. Math. Appl., 12 (1973) 223.
- [4] Ortega J.M., Reinboldt W.C., *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, (1970).
- [5] Grzegórski S.M., *Rescaled multilevel least-change almost secant methods*, „Matematyka Stosowana” XXXIII, (1991).
- [6] Grzegórski S.M., *Orthogonal projections on convex sets for Newton-like methods*, SIAM J.Numer. Anal., 22 (1985) 1208.