



## The shortest path problem

Andrzej Baran\*

*Institute of Physics, Maria Curie-Skłodowska University,  
 Pl. M.Curie-Skłodowskiej 1, 20-031 Lublin, Poland*

### Abstract

The shortest path problem is solved and applied to the calculations of half lives with respect to spontaneous fission of heavy nuclei. The dynamical programming method of Bellman and Kalaba is used to find the fission path in the  $d$ -dimensional space of deformation parameters. Fission half lives of heavy isotopes ( $100 < Z < 110$ ) are shown and compared to the experimental data.

### 1. Introductory remarks

A half life of a nucleus  ${}^A X_Z$  with respect to the spontaneous fission into two nearly equal fragments is inversely proportional to the probability of fission and reads [1]

$$T : \frac{\sim 1}{nP}, \quad (1)$$

where

$$P = \left(1 + \exp(2S)\right)^{-1}, \quad (2)$$

is the probability of fission decay,  $n$  is a frequency of fission mode equal to  $10^{20,38} \text{sec}^{-1}$ ,  $S$  is the functional of path  $\bar{x}$  in the  $d$ -dimensional space of nuclear deformations  $\{x\}$  and is given in units of Plank constant  $h=2p$ . The path  $\bar{x}$  joins two special points of the deformation space: one corresponding to the ground state of the nucleus ( $a$ ) and the other one to the elongated nuclear shape or "two nuclear fragments" ( $b$ ). The functional  $S$  is given by

$$S[x] = \int_{(a)}^{(b)} \sqrt{2|V(x) - E|} B(s) ds, \quad (3)$$

where  $V(s)$  is a potential energy of the nucleus,  $E$  its actual energy and

$$B(s) = \sum_{k,l} B_{kl}(x) \&\&. \quad (4)$$

---

\* E-mail address: [baran@tytan.umcs.lublin.pl](mailto:baran@tytan.umcs.lublin.pl)

The dot over the symbol means the derivative  $\dot{x}(s) = dx/ds$ , and  $s$  is e.g. the arc length of the curve  $x$ . The shortest half life  $T$  (or the largest tunneling probability  $P : 1/T$ ) corresponds to the smallest value of the action  $S$ . The problem of finding the half life is thus equivalent to the minimization problem of the action integral  $S[x]$  in Eq. (3) where  $x$  is the path searched for.

In practice one has values of  $V(x)$  and  $B_{kl}(x)$  in discrete points of  $x$  which can be arbitrarily dense in the sense of  $S$ . If the mesh has dimensions  $\{n_1, n_2, \dots, n_d\}$  then the number of possible paths which have to be searched for is approximately given by (see Fig. 1)

$$(n_1 n_2 n_3 \dots n_d)^m \tag{5}$$

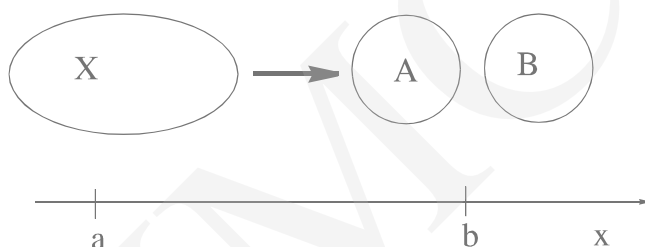


Fig. 1. Spontaneous fission of atomic nucleus X into two fragments A and B

Even in the simplest case of 2-dimensional space (a surface) consisting of e.g.,  $10 \times 20$  grid points this is a tremendously large number. On the first sight the problem can not be solved!

However, if one applies the dynamical programming, the solution of the problem is easy [2]. The method has been applied for the first time in a series of papers on the nuclear fission [1] at the end of 70-s and allowed to determine the half lives of the heaviest and superheavy nuclei.

The presented paper describes the algorithm of finding the shortest path which minimizes the action integral  $S[x]$  given in Eq. (3).

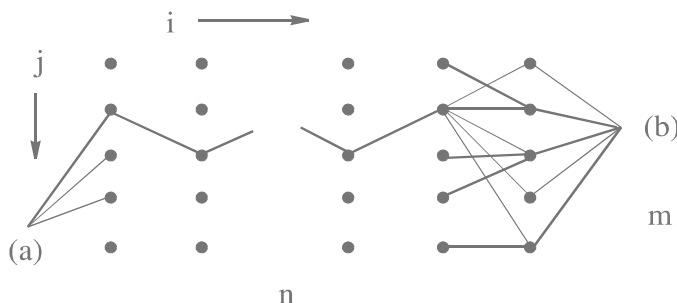


Fig. 2. The shortest path searched by dynamic programming method. The grid points represent the mesh in a 2-dimensional space  $\{x\}$  in which the functional  $S[\bar{x}]$  of Eq. (3) is defined.

The thick polygonal line represents the shortest path starting from the point  $(i, j)$  and ending at the final point  $(b)$

## 2. The method of dynamical programming

*Assumption:* The path joining the two points which are “very close” in the the sense of  $S$  is the straight line. In this case any path on the mesh is the polygonal line.

Figure 1 shows the two dimensional mesh of grid points in  $\{x\}$ . Consider the action  $S(i,j;b)$  corresponding to the minimal path from  $(i,j)$  to  $(b)$  and the action  $S(i-1,k;i,j)$  calculated between the two neighbour points  $(i-1,k)$  and  $(i,j)$  (here  $i$  denotes a column and  $j, k$  the rows of a regular grid). Adding them together and minimizing over all the rows  $k$  one obtains the action

$$S(i-1, j; b) = \min_k [S(i-1, j) + S(i, j; k)], \quad (6)$$

which is the minimal action on the polygonal path joining  $(i-1, j)$  and  $(b)$ . Repeating this procedure for all columns and all points in a given column one ends with the actions  $S(1, k)$  for  $k=1, 2, \dots, m$ . Now, the last step is to calculate the minimum of the expression

$$S(a, k) + S(1, k; b), \quad (7)$$

with respect to  $k$ . If this is done one ends with the shortest path joining  $(a)$  and  $(b)$ . It is really simple!

What was done, can be summarized in the following formula:

$$S(a, b) = \min_{k_1} [S(a; 1, k_1) + \min_{k_2} [S(1, k_1; 2, k_2) + \dots \min_{k_n} [S(n-1, k_n) + S(n, k_n; b)]]]. \quad (8)$$

In this way the problem of complexity  $O(n^m)$  is reduced to the problem of complexity  $O(n \times m)$ . In addition, no part of calculations was repeated. In the case of many dimensional space, the algorithm is really very effective.

The calculations of the half lives by using this method are very common now. The method may be also applied in other branches of science where the minimization of functionals in complex situations is needed.

The convergence of the method presented here was tested by increasing the number of grid points and/or by comparison the path to the shortest path obtained by application of the Ritz method in which one assumes the shape of the path in the form e.g.,  $\bar{x} = \sum_k a_k f(a_1, a_2, \dots, a_N)$ , where  $f(a)$  are properly chosen functions and one looks for the minimum of the function  $S(a_1, a_2, \dots, a_N)$ .

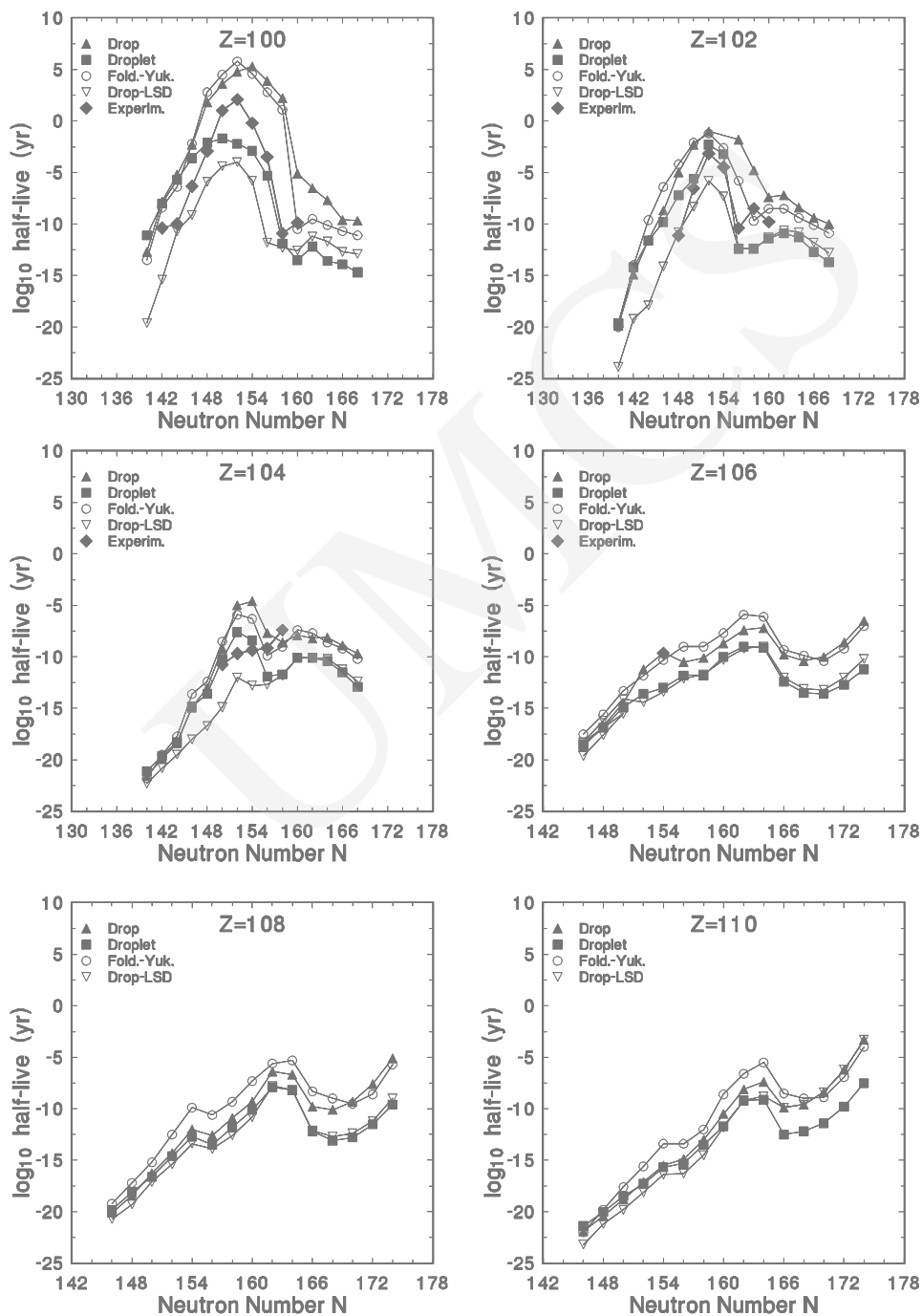


Fig. 3. Spontaneous fission half-lives ( $T_{sf}$ ) (in years) for the even-even isotopes of atomic nuclei with  $100 \leq Z \leq 110$  plotted as a function of the neutron number  $N$ . The different data are displayed for four physical models of the nuclear energy

### 3. Half life examples

Figure 1 shows an example of spontaneous fission half lives of heavy nuclei with the atomic number  $100 \leq Z \leq 110$  calculated by the dynamical programming method [3] in  $d=3$  dimensional space of shape parameters. The experimental data are used for comparison. To calculate the potential energy  $V(x)$  we have used the Woods-Saxon potential with the *optimal* set of parameters. The mass parameters  $B_{ij}$  were calculated in the framework of cranking model.

The results are obtained using four different models for the nuclear energy. The data obtained in the liquid drop model (Drop) [4] are presented by full triangles and the results obtained with the droplet model [5] (Droplet) by squares. The estimates made with the LSD [6] (Drop-LSD) are marked by open triangles. It is seen that the spontaneous fission half-lives differ considerably depending on the model used. For the liquid drop and the folded Yukawa model [7] (Fold.-Yuk.) the results are too large as compared to experiment, while these for the droplet and the LSD models are closer to the measured  $T_{sf}$  values.

For a long time this kind of programming has served as a good recipe which allows to predict the half lives of heavy or superheavy nuclei artificially produced in a few nuclear laboratories in the World, namely in the Lawrence Berkeley National Laboratory, Berkeley, US, the Joint Institute of Nuclear Research (JINR), Dubna, Russia and Gesellschaft für Schwerionen, Darmstadt, Germany.

### 4. Summary

The effective method of dynamical programming is described. It leads to a very effective algorithm of minimization of the functionals defined on the mesh in many dimensional space.

The method of dynamical programming is very cheap and easy to apply. As an example of application of the method we show the results of calculations of spontaneous fission half lives of the heaviest nuclei  $Z \geq 100$ .

### References

- [1] Baran A., Pomorski K., Łukasiak A., Sobiczewski, A., *A dynamical analysis of spontaneous-fission half-lives*. Nucl. Phys., A361 (1981) 83.
- [2] Bellman R.E., Kalaba R.E., *Quasilinearization and Nonlinear Boundary Value Problems*. American Elsevier Publ. Comp. Inc., New York, (1965).
- [3] Łojewski Z., Baran A., Pomorski K., *Acta Physica Pol.*, to be published, (2003).
- [4] Myers W.D., Swiatecki W.J., Nucl. Phys. 81, (1966), 1, Ark. Fys., 36 (1967) 343.
- [5] Myers W.D., Swiatecki W.J., Ann. Phys. 84 (1974) 186.
- [6] Pomorski K., Dudek J., *Nuclear liquid drop model with the surface-curvature terms*. Phys. Rev.C, submitted for publication,
- [7] Krappe H.J., Nix J.R., Sierk A.J., Phys.Rev., C20 (1979) 992.